DEGREE OF PARABOLIC QUANTUM GROUPS

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Abstract. We study some elementary properties of the quantum enveloping algebra associated to a parabolic subalgebra $\mathfrak{p}$ of a semisimple Lie algebra $\mathfrak{g}$. In particular we prove an explicit formula for the degree of this algebra, that extends the well known formula for the quantum enveloping algebra associated to $\mathfrak{g}$ and $\mathfrak{b}$, where $\mathfrak{b}$ is a Borel subalgebra of $\mathfrak{g}$.

1. Introduction and overview

The aim of this work is to calculate the degree of some quantum universal enveloping algebras. Let $\mathfrak{g}$ be a semisimple Lie algebra, fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a Borel subalgebra $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$, and denote with Π the correspondent set of simple roots. Given $\Pi' \subset \Pi$, we associate to Π' the parabolic subalgebra $\mathfrak{p} \supset \mathfrak{b}$. In this situation we can "quantized" our algebras, we obtain Hopf algebras over $\mathbb{C}\left[q, q^{-1}\right]$, namely $\mathcal{U}_q(\mathfrak{b}) \subset \mathcal{U}_q(\mathfrak{p}) \subset \mathcal{U}_q(\mathfrak{g})$.

When we specialize the parameter q to a primitive l^{th} root ϵ of 1 (with some restrictions on l), the resulting algebras are finite modules over their centers, and they are a finitely generated $\mathbb{C}$ algebra. In particular, every irreducible representations has finite dimension. Let us denote by $\mathcal{V}$ the set of irreducible representations, Schur lemma gives us a surjective application

$$\pi : \mathcal{V} \to \operatorname{Spec}(Z).$$

To determine the pull back of a point in $\operatorname{Spec}(Z)$ is a very difficults problem. But generically the problem becomes easier. Since our algebras are domains, there exists a non empty Zariski open set $V \subset \operatorname{Spec}(Z)$, such that $\pi|_{\pi^{-1}(V)}$ is bijective and moreover every irreducible representation in $\pi^{-1}(V)$ has the same dimension d, the degree of our algebra. The problem is to identify d.

Note that, a natural candidate for d exists. We will see that in the case of $\mathcal{U}_\epsilon(\mathfrak{p})$, we can find a natural subalgebra $Z_0 \subset Z$, which is a Hopf subalgebra of $\mathcal{U}_\epsilon(\mathfrak{p})$. Therefore it is the coordinate ring of an algebraic group H. The deformation structure of $\mathcal{U}_\epsilon(p)$ implies that H has a Poisson structure. Let δ be the maximal dimension of the symplectic leaves, then a natural conjecture is $d = l^{\frac{\delta}{2}}$. This is well know in several cases, for example, $\mathfrak{p} = \mathfrak{g}$ and $\mathfrak{p} = \mathfrak{b}$ (cf [DCK90] and [KW76]).

Our purpose has been to prove one explicit formula for δ. Before describing the strategy of the proof, we explain the formula for δ. Set $\mathfrak{l}$ the Levi factor of $\mathfrak{p}$. Let $\mathcal{W}$ be the Weyl group of the root system of $\mathfrak{g}$, and $\mathcal{W}^{\mathfrak{l}} \subset \mathcal{W}$ that one of subsystem generated by Π'. Denote by $\mathbf{w}_0$ the longest element of $\mathcal{W}$ and $\mathbf{w}_0^{\mathfrak{l}}$ the

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longest element of $\mathcal{W}^{\mathfrak{l}}$. Recall that $\mathcal{W}$ acts on $\mathfrak{h}$ and set s as the rank of the linear transformation $w_0 - w_0^{\mathfrak{l}}$ of $\mathfrak{h}$. Then

$$\delta = l(w_0) + l(w_0^{\mathfrak{l}}) + s,$$

where l is the length function with respect to the simple reflection. We describe now the strategy of the proof. Our instrument is the theory of quasi polynomial algebras. By a result of De Concini, Kaç and Procesi ([DCKP92] and [DCKP95]), we know that in order to compute the degree of such algebras we can be reduced to the computation of the rank of a skew symmetric matrix. To use this result we construct a degeneration of our quantum algebras to a quasipolynomial algebra, $\mathcal{U}_\epsilon^t$, and a family, $\mathcal{U}_\epsilon^{t,\chi}$, of finitely generated algebras parameterized by $(t,\chi) \in \mathbb{C} \times \mathrm{Spec}(Z_0)$. Then we prove that U_ϵ^0 is a quasi polynomial algebra so that the theorem of De Concini, Kaç and Procesi ([DCKP95]) can be applied but we notice that this give us only a lower bound for the value of d. To get the equality we use the family $\mathcal{U}_\epsilon^{t,\chi}$. The rigidity of the semisimple algebras gives us $\mathcal{U}_\epsilon^{1,\chi} \cong \mathcal{U}_\epsilon^{0,\chi}$ and the theory of the algebras with trace ([DCP93]) it tells us that the degree of our quantum algebra and $\mathcal{U}_\epsilon^{1,\chi}$ are equal, and that the same is true for the quasipolynomial algebra $\mathcal{U}_\epsilon^0$ and $\mathcal{U}_\epsilon^{0,\chi}$, then this give the desired deduction. We close this introduction with the description of the section that compose this article. In the first section we introduce the main object of this work the quantum universal enveloping algebra associated to a parabolic Lie algebra and we give some elementary properties for this algebras. The next two sections are dedicated to the proof of the formula for the degree, in particular in section 3 we describe the main tool of this work, the degeneration of our quantum algebra to a quasi polynomial algebra. The last section is devoted to the study of the center of the deformation, note that the actual determination of the center of the algebra $\mathcal{U}_\epsilon(\mathfrak{p})$ remains in general an open and potentially tricky problem. However we will propose a method, inspired by work of Premet and Skryabin ([PS99]), to "lift" elements of the center of the degenerate algebra at $t = 0$ to elements of the center at least over an open set of $\mathrm{Spec}\, Z_0$, and we prove that the center is deformation invariant.

2. Quantum enveloping algebras

We begin by recalling some classical facts about quantum enveloping algebras associated to a simple Lie algebra $\mathfrak{g}$, and we introduce the main object of this work, the quantum enveloping algebra associated to a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$.

Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. Let $R \supset R^+ \supset \Pi = \{\alpha_1, \ldots, \alpha_n\}$ be the root system, set of positive root and set of simple root. As usual we call $\mathcal{W}$ the Weyl group associated to $\mathfrak{h}$ and $\mathfrak{b}$ and we set $\mathbf{w}_0$ as the longest element of $\mathcal{W}$. Denote by P and Q the its weight and root lattices and let $w_1, \ldots, w_n$ be the fundamental weight.

Fix $\Pi' \subset \Pi$, we call $\mathfrak{p}$ the parabolic subalgebra associated to it. Note that if $\Pi' = \emptyset$ then $\mathfrak{p} = \mathfrak{b}$ and if $\Pi' = \Pi$ then $\mathfrak{p} = \mathfrak{g}$. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be the Levi decomposition of $\mathfrak{p}$, with $\mathfrak{l}$ the Levi factor and $\mathfrak{u}$ the unipotent part. We call $\mathcal{W}^{\mathfrak{l}} \subset \mathcal{W}$ the Weyl group of $\mathfrak{l}$ and $\mathbf{w}_0^{\mathfrak{l}}$ its longest element, and we have $\Pi' = \Pi^{\mathfrak{l}}$.

2.1. **The simple case.**

Following De Concini and Kac ([DCK90]), we define:

Definition 1. *A* simply connected quantum group $\mathcal{U}_q(\mathfrak{g})$ *associated to the Cartan matrix* $C = (c_{i,j})_{i,j=1,\ldots,n}$ *is an algebra over* $\mathbb{C}(q)$ *on generators* E_i*,* F_i *(*$i =$

$1, \cdots, n)$, K_α $\alpha \in P$, subject to the following relations

$$\begin{cases} K_\alpha K_\beta = K_{\alpha+\beta} \\ K_0 = 1 \end{cases} \tag{2.1}$$

$$\begin{cases} K_\alpha E_i K_{-\alpha} = q^{(\alpha|\alpha_i)} E_i \\ K_\alpha F_i K_{-\alpha} = q^{-(\alpha|\alpha_i)} F_i \end{cases} \tag{2.2}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q^{d_i} - q^{-d_i}} \tag{2.3}$$

$$\begin{cases} \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^{1-a_{ij}-s} E_j E_i^s = 0 \text{ if } i \neq j \\ \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s} F_j F_i^s = 0 \text{ if } i \neq j \end{cases} \tag{2.4}$$

where $\begin{bmatrix} n \\ m \end{bmatrix}_{d_i}$ is the q binomial coefficient defined by:

$$\begin{bmatrix} n \\ m \end{bmatrix}_d = \frac{[n]_d!}{[k]_d![n-k]_d!}$$

and

$$[n]_d = \frac{q^n - q^{-n}}{q^d - q^{-d}}.$$

It is well known, by the work of Lusztig ([Lus93]), that

Theorem 1. *$\mathcal{U}_q(\mathfrak{g})$ has a Hopf algebra structure with comultiplication Δ, antipode S and counit η defined by:*

- $\begin{cases} \Delta(E_i) = E_i \otimes 1 + K_{\alpha_i} \otimes E_i \\ \Delta(F_i) = F_i \otimes K_{-\alpha_i} + 1 \otimes F_i \\ \Delta(K_\alpha) = K_\alpha \otimes K_\alpha \end{cases}$
- $\begin{cases} S(E_i) = -K_{\alpha_i} E_i \\ S(F_i) = -F_i K_{\alpha_i} \\ S(K_\alpha) = K_{-\alpha} \end{cases}$
- $\begin{cases} \eta(E_i) = 0 \\ \eta(F_i) = 0 \\ \eta(K_\alpha) = 1 \end{cases}$

We denote by $\mathcal{U}^+$, $\mathcal{U}^-$ and $\mathcal{U}^0$ the $\mathbb{C}(q)$-subalgebra generated by the E_i, the F_i and K_β respectively. The algebras $\mathcal{U}^+$ and $\mathcal{U}^-$ are not Hopf subalgebras. On the other hand, the algebras $\mathcal{U}^{\geq 0} := \mathcal{U}^+\mathcal{U}^0$ and $\mathcal{U}^{\leq 0} := \mathcal{U}^0\mathcal{U}^-$ are Hopf subalgebras and we shall think to them as quantum deformation of the enveloping algebras $\mathcal{U}(\mathfrak{b})$ and $\mathcal{U}(\mathfrak{b}^-)$. We denote them $\mathcal{U}_q(\mathfrak{b})$ and $\mathcal{U}_q(\mathfrak{b}^-)$.

Following Lusztig ([Lus93]), we define an action of the braid group $\mathcal{B}_\mathcal{W}$ (associated to $\mathcal{W}$). Denote by T_i and s_i the canonical generators of $\mathcal{B}_\mathcal{W}$ and $\mathcal{W}$, we define the action as an automorphism of $\mathcal{U}_{\mathfrak{q}}(\mathfrak{g})$, by the formulas:

$$T_i K_\lambda = K_{s_i(\lambda)} \tag{2.5}$$

$$T_i E_i = -F_i K_i \tag{2.6}$$

$$T_i F_i = -K_i^{-1} E_i \tag{2.7}$$

$$T_i E_j = \sum_{s=0}^{-c_{ij}} (-1)^{s-c_{ij}} q^{-sd_i} \frac{E_i^{-c_{ij}-s}}{[-c_{ij}-s]_{d_i}!} E_j \frac{E_i^s}{[s]_{d_i}!} \tag{2.8}$$

$$T_i F_j = \sum_{s=0}^{-c_{ij}} (-1)^{s-c_{ij}} q^{sd_i} \frac{F_i^s}{[s]_{d_i}!} F_j \frac{F_i^{-c_{ij}-s}}{[-c_{ij}-s]_{d_i}!}. \tag{2.9}$$

We use the braid group to construct analogues of the root vectors associated to non simple roots.

Take a reduced expression $\mathbf{w}_0 = s_{i_1} \dots s_{i_N}$ for the longest element in the Weyl group $\mathcal{W}$. Setting $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_j)$, we get a total order on the set of positive root. We define the elements $E_{\beta_j} = T_{i_1} \dots T_{i_{j-1}}(E_{i_j})$ and $F_{\beta_j} = T_{i_1} \dots T_{i_{j-1}}(F_{i_j})$. Note that this elements depend on the choice of the reduced expression.

Lemma 1. *(i) $E_{\beta_j} \in \mathcal{U}^+$, $\forall\ i = 1 \dots N$ and the monomials $E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N}$ form a $\mathbb{C}(q)$ basis of $\mathcal{U}^+$*

(ii) $F_{\beta_j} \in \mathcal{U}^-$, $\forall\ i = 1 \dots N$ and the monomials $F_{\beta_1}^{k_1} \cdots F_{\beta_N}^{k_N}$ form a $\mathbb{C}(q)$ basis of $\mathcal{U}^-$

Theorem 2 (Poincaré-Birkoff-Witt theorem)**.** *The monomials*

$$E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N} K_\alpha F_{\beta_N}^{k_N} \cdots F_{\beta_1}^{k_1}$$

are a $\mathbb{C}(q)$ basis of $\mathcal{U}$. In fact as vector spaces, we have the tensor product decomposition,

$$\mathcal{U} = \mathcal{U}^+ \otimes \mathcal{U}^0 \otimes \mathcal{U}^-$$

Proof. See [Lus93]. □

Theorem 3 (Levendorskii-Soibelman relations)**.** *For $i < j$ one has*

(i)

$$E_{\beta_j} E_{\beta_i} - q^{(\beta_i|\beta_j)} E_{\beta_i} E_{\beta_j} = \sum_{k \in \mathbb{Z}_+^N} c_k E^k \tag{2.10}$$

where $c_k \in \mathbb{C}[q, q^{-1}]$ and $c_k \neq 0$ only when $k = (k_1, \dots, k_N)$ is such that $k_s = 0$ for $s \leq i$ and $s \geq j$, and $E^k = E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N}$.

(ii)

$$F_{\beta_j} F_{\beta_i} - q^{-(\beta_i|\beta_j)} F_{\beta_i} F_{\beta_j} = \sum_{k \in \mathbb{Z}_+^N} c_k F^k \tag{2.11}$$

where $c_k \in \mathbb{C}[q, q^{-1}]$ and $c_k \neq 0$ only when $k = (k_1, \cdots, k_N)$ is such that $k_s = 0$ for $s \leq i$ and $s \geq j$, and $F^k = F_{\beta_N}^{k_N} \cdots F_{\beta_1}^{k_1}$.

Proof. See [LS91]. □

To obtain from $\mathcal{U}_q(\mathfrak{g})$ a well defined Hopf algebra by specializing q to an arbitrary non zero complex number ϵ, one can construct an *integral form* of $\mathcal{U}$.

Definition 2. *An* integral form $\mathcal{U}_{\mathcal{A}}$ *is a* $\mathcal{A}$ *subalgebra, where* $\mathcal{A} = \mathbb{C}[q, q^{-1}]$*, such that the natural map*

$$\mathcal{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}(q) \mapsto \mathcal{U}$$

is an isomorphism of $\mathbb{C}(q)$ *algebra. We define*

$$\mathcal{U}_{\epsilon} = \mathcal{U}_{\mathcal{A}} \otimes_{\mathcal{A}} \mathbb{C}$$

using the homomorphism $\mathcal{A} \mapsto \mathbb{C}$ *mapping* q *to* ϵ*.*

There are two different candidates for $\mathcal{U}_{\mathcal{A}}$ the non restricted and the restricted integral form, which lead to different specializations (with markedly different representation theories) for certain values of ϵ. We are interested in the non restricted form, for more details one can see [CP95].

Introduce the elements

$$[K_i; m]_{q_i} = \frac{K_i q_i^m - K_i^{-1} q_i^{-m}}{q_i - q_i^{-1}} \in \mathcal{U}^0$$

with $m \geq 0$, where $q_i = q^{d_i}$.

Definition 3. *The algebra* $\mathcal{U}_{\mathcal{A}}$ *is the* $\mathcal{A}$ *subalgebra of* $\mathcal{U}$ *generated by the elements* E_i*,* F_i*,* $K_i^{\pm 1}$ *and* $L_i = [K_i; 0]_{q_i}$*, for* $i = 1, \dots, n$*. With the map* Δ*,* S *and* η *defined on the first set of generators as in 1 and with*

$$\Delta(L_i) = L_i \otimes K_i + K_i^{-1} \otimes L_i \tag{2.12a}$$
$$S(L_i) = -L_i \tag{2.12b}$$
$$\eta(L_i) = 0 \tag{2.12c}$$

The defining relation of $\mathcal{U}_{\mathcal{A}}$ are as in 1 replacing 2.3 by

$$E_i F_j - F_j E_i = \delta_{ij} L_i$$

and adding the relation

$$(q_i - q_i^{-1}) L_i = K_i - K_i^{-1}$$

Proposition 1. $\mathcal{U}_{\mathcal{A}}$ *with the previous definition is a Hopf algebra. Moreover,* $\mathcal{U}_{\mathcal{A}}$ *is an integral form of* $\mathcal{U}$*.*

Proof. See [CP95] or [DCP93] §12. □

Proposition 2. *If* $\epsilon^{2d_i} \neq 1$ *for all* i*, then*

(i) $\mathcal{U}_{\epsilon}$ *is generated over* $\mathbb{C}$ *by the elements* E_i*,* F_i*, and* $K_i^{\pm 1}$ *with defining relations obtained from those in 1 by replacing* q *by* ϵ

(ii) The monomials

$$E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N} K_{\alpha} F_{\beta_N}^{k_N} \cdots F_{\beta_1}^{k_1}$$

form a $\mathbb{C}$ *basis of* $\mathcal{U}_{\epsilon}$*.*

(iii) The LS relations holds in $\mathcal{U}_{\epsilon}$*.*

Proof. See [DCP93] §12. □

2.2. **Parabolic case.** Choose a reduced expression of $w_0 = s_{j_1} \dots s_{j_k} s_{i_1} \dots s_{i_h}$, such that $w_0^{\mathfrak{l}} = s_{i_1} \dots s_{i_h}$ is a reduced expression for $w_0^{\mathfrak{l}}$. Set $\overline{w} = w_0 (w_0^{\mathfrak{l}})^{-1} = s_{j_1} \dots s_{j_k}$, with $h = |(R^{\mathfrak{l}})^+|$ and $h + k = N = |R^+|$. Define, as in the general case,

$$\begin{aligned} \beta_t^1 &= \overline{w} s_{i_1} \dots s_{i_{t-1}} (\alpha_{i_t}) \in (R^{\mathfrak{l}})^+, \\ \beta_t^2 &= s_{j_1} \dots s_{j_{t-1}} \left(\alpha_{i_{t+k}}\right) \in R^+ \setminus (R^{\mathfrak{l}})^+. \end{aligned}$$

Given this choice of positive root, we obtain the following q analogues of the root vectors:

$$\begin{aligned} E_{\beta_t^1} &= T_{\overline{w}} T_{i_1} \dots T_{i_{t-1}} (E_{i_t}), \\ E_{\beta_t^2} &= T_{j_1} \dots T_{j_{t-1}} \left(E_{i_{t+k}}\right). \end{aligned}$$

and

$$\begin{aligned} F_{\beta_t^1} &= T_{\overline{w}} T_{i_1} \dots T_{i_{t-1}} (F_{i_t}), \\ F_{\beta_t^2} &= T_{j_1} \dots T_{j_{t-1}} \left(F_{i_{t+k}}\right). \end{aligned}$$

The PBW theorem implies that the monomials

$$E_{\beta_1^2}^{s_1} \cdots E_{\beta_k^2}^{s_k} E_{\beta_1^1}^{s_{k+1}} \cdots E_{\beta_h^1}^{s_{k+h}} K_\lambda F_{\beta_h^1}^{t_{k+h}} \cdots F_{\beta_1^1}^{t_{k+1}} F_{\beta_k^2}^{t_k} \cdots F_{\beta_1^2}^{t_1} \tag{2.13}$$

for $(s_1, \cdots, s_N)$, $(t_1, \dots, t_N) \in (\mathbb{Z}^+)^N$ and $\lambda \in \Lambda$, form a $\mathbb{C}(q)$-basis of $\mathcal{U}_q(\mathfrak{g})$.

The choice of the reduced expression of w_0 and the LS relations for $\mathcal{U}_q(\mathfrak{g})$ implies that

Proposition 3. *For $i < j$ one has*

(i)

$$E_{\beta_j^1} E_{\beta_i^1} - q^{(\beta_i^1|\beta_j^1)} E_{\beta_i^1} E_{\beta_j^1} = \sum_{k \in \mathbb{Z}_+^N} c_k E_1^k$$

where $c_k \in \mathbb{C}(q)$ and $c_k \neq 0$ only when $k = (s_1, \dots, s_k)$ is such that $s_r = 0$ for $r \leq i$ and $r \geq j$, and $E_1^k = E_{\beta_1^1}^{s_1} \dots E_{\beta_k^1}^{s_k}$.

(ii)

$$E_{\beta_j^2} E_{\beta_i^2} - q^{-(\beta_i^2|\beta_j^2)} E_{\beta_i^2} E_{\beta_j^2} = \sum_{k \in \mathbb{Z}_+^N} c_k E_2^k$$

where $c_k \in \mathbb{C}(q)$ and $c_k \neq 0$ only when $k = (t_1, \dots, t_h)$ is such that $t_r = 0$ for $r \leq i$ and $r \geq j$, and $E_2^k = E_{\beta_1^2}^{t_1} \dots E_{\beta_h^2}^{t_h}$.

The same statement holds for $F_{\beta_i^1}$ and $F_{\beta_i^2}$.

Let $\Pi^{\mathfrak{l}}$ be the simple root associated to the Levi factor $\mathfrak{l}$. The definition of the braid group action implies:

Proposition 4. *(i) If $i \in \Pi^{\mathfrak{l}}$, then $E_i = E_{\beta_s^1}$ and $F_i = F_{\beta_s^1}$, for some $s \in \{1, \dots, h\}$.*

(ii) If $i \in \Pi \setminus \Pi^{\mathfrak{l}}$, then $E_i = E_{\beta_s^2}$ and $F_i = F_{\beta_s^2}$ for some $s \in \{1, \dots, k\}$.

Definition 4. *The simple connected quantum group associated to $\mathfrak{p}$, or* parabolic quantum group, *is the $\mathbb{C}(q)$ subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by*

$$\mathcal{U}_q(\mathfrak{p}) = \langle E_{\beta_i^1}, K_\lambda, F_{\beta_j} \rangle$$

for $i = 1, \dots, h$, $j = 1 \dots N$ and $\lambda \in \Lambda$.

Definition 5. *(1) The quantum Levi factor of $\mathcal{U}_q(\mathfrak{p})$ is the subalgebra generated by*

$$\mathcal{U}_q(\mathfrak{l}) = \langle E_{\beta_i^1}, K_\lambda, F_{\beta_i^1} \rangle$$

for $i = 1, \dots, h$, and $\lambda \in \Lambda$.

(2) The quantum unipotent part of $\mathcal{U}(\mathfrak{p})$ is the subalgebra generated by

$$\mathcal{U}^{\overline{w}} = \langle F_{\beta_s^2} \rangle$$

with $s = 1 \dots h$

Set $\mathcal{U}_q^+(\mathfrak{p}) = \mathcal{U}_q^+(\mathfrak{l}) = \langle E_i \rangle_{i \in \Pi^{\mathfrak{l}}}$, $\mathcal{U}_q^-(\mathfrak{p}) = \langle F_i \rangle_{i \in \Pi}$, $\mathcal{U}_q^-(\mathfrak{l}) = \langle F_i \rangle_{i \in \Pi^{\mathfrak{l}}}$ and $\mathcal{U}_q^0(\mathfrak{p}) = \mathcal{U}_q^0(\mathfrak{l}) = \langle K_\lambda \rangle_{\lambda \in \Lambda}$. We have:

Proposition 5. *The definition of $\mathcal{U}_q(\mathfrak{p})$ and $\mathcal{U}_q(\mathfrak{l})$ is independent of the choice of the reduced expression of $w_0^{\mathfrak{l}}$ and w_0.*

Proof. Follows immediately from proposition 9.3 in [DCP93]. □

We can easely see taht the PBW theorem and the LS relations holds in $\mathcal{U}_q(\mathfrak{p})$ and $\mathcal{U}_q(\mathfrak{l})$, which is an immediately consequence of 2.13.

Proposition 6. *Set $m = \operatorname{rank} \mathfrak{l} = \#|\Pi^{\mathfrak{l}}|$. The algebra $\mathcal{U}(\mathfrak{p})$ is generated by E_i, F_j K_λ, with $i = 1, \dots, m$, $j = 1, \dots, n$ and $\lambda \in \Lambda$, subject to the following relations:*

$$\begin{cases} K_\alpha K_\beta = K_{\alpha+\beta} \\ K_0 = 1 \end{cases} \tag{2.14}$$

$$\begin{cases} K_\alpha E_i K_{-\alpha} = q^{(\alpha|\alpha_i)} E_i \\ K_\alpha F_j K_{-\alpha} = q^{-(\alpha|\alpha_j)} F_j \end{cases} \tag{2.15}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q^{d_i} - q^{-d_i}} \tag{2.16}$$

$$\begin{cases} \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} E_i^{1-a_{ij}-s} E_j E_i^s = 0 \ \textit{if } i \neq j \\ \sum_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s} F_j F_i^s = 0 \ \textit{if } i \neq j. \end{cases} \tag{2.17}$$

Where $\begin{bmatrix} n \\ m \end{bmatrix}_{d_i}$ is the q binomial coefficient.

Proof. Follows from PBW theorem and the LS relations. □

We state now some easy properties of $\mathcal{U}_q(\mathfrak{p})$:

Lemma 2. *(i) The multiplication map*

$$\mathcal{U}^+(\mathfrak{l}) \otimes \mathcal{U}^0(\mathfrak{l}) \otimes \mathcal{U}^-(\mathfrak{l}) \to \mathcal{U}(\mathfrak{l})$$

is an isomorphism of vector spaces.

(ii) The multiplication map

$$\mathcal{U}(\mathfrak{l}) \otimes \mathcal{U}^{\overline{w}} \xrightarrow{m} \mathcal{U}(p)$$

defined by $m(x, u) = xu$ for every $x \in \mathcal{U}(\mathfrak{l})$ and $u \in \mathcal{U}^{\overline{w}}$, is an isomorphism of vector spaces.

(iii) *The map* $\mu : \mathcal{U}(\mathfrak{p}) \to \mathcal{U}(\mathfrak{l})$ *defined by*

$$\mu\left(E^{s_1}_{\beta^1_1}\cdots E^{s_h}_{\beta^1_h}K_\lambda F^{t_{k+h}}_{\beta^1_h}\cdots F^{t_{k+1}}_{\beta^1_1}F^{t_k}_{\beta^2_k}\cdots F^{t_1}_{\beta^2_1}\right)$$
$$= \begin{cases} 0 & \text{if } t_{k+i}\neq 0 \text{ for some } i=1,\dots,h,\\ E^{s_1}_{\beta^1_1}\cdots E^{s_h}_{\beta^1_h}K_\lambda F^{t_{k+h}}_{\beta^1_h}\cdots F^{t_{k+1}}_{\beta^2_1} & \text{if } t_{k+i}=0 \text{ for all } i=1,\dots,h,\end{cases}$$

is an homomorphism of algebras.

(iv) $\mathcal{U}(\mathfrak{p})$ *and* $\mathcal{U}(\mathfrak{l})$ *are Hopf subalgebras of* $\mathcal{U}$.

Proof. Follows immediately from the definition. □

Let $\mathcal{A} = \mathbb{C}[q, q^{-1}]$, and $\mathcal{U}_\mathcal{A}$ the integral form of $\mathcal{U}_q(\mathfrak{g})$ defined in definition 3. Like in the general case, we define $\mathcal{U}_\mathcal{A}(\mathfrak{p})$, has the subalgebra generated by $E_{\beta^1_i}$, $F_{\beta^1_i}$, $F_{\beta^2_s}$, $K^{\pm1}_j$ and L_j, with $i = 1,\dots,h$, $s = 1,\dots,k$ and $j = 1,\dots,n$.

Definition 6. *Let* $\epsilon \in \mathbb{C}$, *we define*

$$\mathcal{U}_\epsilon(\mathfrak{p}) = \mathcal{U}_\mathcal{A}(\mathfrak{p}) \otimes_\mathcal{A} \mathbb{C}$$

using the homomorphism $\mathcal{A} \to \mathbb{C}$ *mapping* $q \to \epsilon$

Let $\epsilon \in \mathbb{C}$ such that $\epsilon^{2d_i} \neq 1$ for all i, then

Proposition 7. $\mathcal{U}_\epsilon(\mathfrak{p}) \subset \mathcal{U}_\epsilon(\mathfrak{g})$. *Moreover* $\mathcal{U}_\epsilon(\mathfrak{p})$ *is generated by* $E_{\beta^1_i}$, F_{β_s} *and* $K^{\pm1}_j$, *for* $i = 1,\dots,h$, $s = 1,\dots,N$ *and* $j = 1,\dots,n$.

Proof. The claim is a consequence of the definition of $\mathcal{U}_\mathcal{A}(\mathfrak{p})$. □

Proposition 8. *The PBW theorem and the LS relations holds for* $\mathcal{U}_\epsilon(\mathfrak{p})$

Proof. The claim is a consequence of the PBW theorem and LS relations for $\mathcal{U}_\epsilon(\mathfrak{g})$ and the choice of the decomposition of the reduced expression of w_0. □

2.3. **Some observations on the center of** $\mathcal{U}_\epsilon(\mathfrak{p})$**.** The aim of this section is to extend some properties of the center of $\mathcal{U}_\epsilon$ at the center of $\mathcal{U}_\epsilon(\mathfrak{p})$.

Proposition 9. *For* $i = 1,\dots,k$, $s = 1,\dots,h$ *and* $j = 1,\dots,n$, $E^l_{\beta^1_i}$, $F^l_{\beta^1_i}$, $F^l_{\beta^2_s}$ *and* $K^{\pm l}_j$ *lie in the center of* $\mathcal{U}_\epsilon(\mathfrak{p})$

Proof. It is well known that these elements lie in the center of $\mathcal{U}_\epsilon$ (cf. [DCP93]), but they also lie in $\mathcal{U}_\epsilon(\mathfrak{p})$, hence the claim. □

For $\alpha \in (R^\mathfrak{l})^+$, $\beta \in R^+$ and $\lambda \in Q$, define $e_\alpha = E^l_\alpha$, $f_\beta = F^l_\beta$, $k^{\pm1}_\lambda = K^{\pm l}_\lambda$. Let $Z_0(\mathfrak{p})$ be the subalgebra generated by the e_α, f_β and $k^{\pm1}_i$.

Proposition 10. *Let* Z^0_0, Z^+_0 *and* Z^-_0 *be the subalgebra generated by* $k^{\pm1}_i$, e_α *and* f_β *respectively.*

(i) $Z^\pm_0 \subset \mathcal{U}^\pm_\epsilon(\mathfrak{p})$

(ii) *Multiplication defines an isomorphism of algebras*

$$Z^-_0 \otimes Z^0_0 \otimes Z^+_0 \to Z_0(\mathfrak{p})$$

(iii) Z^0_0 *is the algebra of Laurent polynomial in the* k_i, *and* Z^+_0 *and* Z^-_0 *are polynomial algebra with generators* e_α *and* f_β *respectively.*

(iv) $\mathcal{U}_\epsilon(\mathfrak{p})$ is a free $Z^0_\epsilon(\mathfrak{p})$ module with basis the set of monomial

$$E^{s_1}_{\beta^1_1}\cdots E^{s_h}_{\beta^1_h}K^{r_1}_1\cdots K^{r_n}_n F^{t_{k+h}}_{\beta^1_h}\cdots F^{t_{k+1}}_{\beta^1_1}F^{t_k}_{\beta^2_k}\cdots F^{t_1}_{\beta^2_1}$$

for which $0\le s_j, t_i, r_v < l$

Proof. By definition of $\mathcal{U}^+(\mathfrak{p})$, we have $e_\alpha \in \mathcal{U}^+(\mathfrak{p})$, since $\mathcal{U}^+(\mathfrak{p})$ is a subalgebra (i) follows. (ii) and (iii) are easy corollaries of the definitions and of the PBW theorem. (iv) follows from the PBW theorem for $\mathcal{U}(\mathfrak{p})$. □

The previous proposition shows that $\mathcal{U}_\epsilon(\mathfrak{p})$ is a finite $Z_0(\mathfrak{p})$ module. Since Z_0 is clearly Noetherian, from (iii), it follows that $Z_\epsilon(\mathfrak{p})\subset\mathcal{U}_\epsilon(\mathfrak{p})$ is a finite $Z_0(\mathfrak{p})$ module, and hence integral over $Z_0(\mathfrak{p})$. By the Hilbert basis theorem $Z_\epsilon(\mathfrak{p})$ is a finitely generated algebra. Thus the affine schemes $\mathrm{Spec}(Z_\epsilon(\mathfrak{p}))$ and $\mathrm{Spec}(Z_0(\mathfrak{p}))$ are algebraic varieties. Note that $\mathrm{Spec}(Z_0)$ is isomorphic to $\mathbb{C}^N\times\mathbb{C}^{l(h)}\times(\mathbb{C}^*)^n$. Moreover the inclusion $Z_0(\mathfrak{p})\hookrightarrow Z_\epsilon(\mathfrak{p})$ induces a projection $\tau:\mathrm{Spec}(Z_\epsilon(\mathfrak{p}))\to\mathrm{Spec}(Z_0(\mathfrak{p}))$, and we have

Proposition 11. $\mathrm{Spec}(Z_\epsilon(\mathfrak{p}))$ *is an affine variety and τ is a finite surjective map.*

Proof. Follows from the Cohen-Seidenberg theorem ([Ser65] ch. III). □

We conclude this section by discussing the relation between the center and the Hopf algebra structure of $\mathcal{U}_\epsilon(\mathfrak{p})$.

Proposition 12. *(i) $Z_0(\mathfrak{p})$ is a Hopf subalgebra of $\mathcal{U}_\epsilon(\mathfrak{p})$.*
(ii) $Z_0(\mathfrak{p})$ is a Hopf subalgebra of Z_0.
(iii) $Z_0(\mathfrak{p}) = Z_0\cap\mathcal{U}_\epsilon(\mathfrak{p})$.

Proof. It follows directly from the given definitions. □

The fact that $Z_0(\mathfrak{p})$ is an Hopf algebra tells us that $\mathrm{Spec}(Z_0(\mathfrak{p}))$ is an algebraic group . Moreover, the inclusion $Z_0(\mathfrak{p})\hookrightarrow Z_0$ being an inclusion of Hopf algebras, induces a group homomorphism,

$$\mathrm{Spec}(Z_0)\to\mathrm{Spec}(Z_0(\mathfrak{p})).$$

Let us recall that in [DCKP92] the authors prove that the center Z_0 has the following form:

$$\mathrm{Spec}(Z_0)=\big\{(a,b):\in B^-\times B^+:\pi^-(a)\pi^+(b)=1\big\}$$

where if we denote by G the connected simply connected Lie group associated to $\mathfrak{g}$, then $B^\pm$ are the borel subgroups of G associated to $\mathfrak{b}^\pm$, H is the toral subgroup associated to $\mathfrak{h}$ and $\pi^\pm: B^\pm\to H$ is the canonical map. From this and, the explicit description of the subalgebra $Z_0(\mathfrak{p})\subset Z_0$, we get

$$\mathrm{Spec}(Z_0(\mathfrak{p}))=\big\{(a,b):\in B^-_L\times B^+:\pi^-(a)\pi^+(b)=1\big\}$$

where $L\subset G$ is the connected subgroup of G such that $Lie(L)=\mathfrak{l}$, and $B^-_L = B^-\cap L$.

3. Degeneration to a quasi polynomial algebra

3.1. The case $\mathfrak{p} = \mathfrak{g}$.

Definition 7. *Let $t \in \mathbb{C}$, we define $\mathcal{U}_\epsilon^t$ the algebra over $\mathbb{C}$ on generators E_i, F_i, L_i and $K_i^{\pm}$, for $i = 1, \ldots, n$, subject to the following relations:*

$$\begin{cases} K_i^{\pm 1} K_j^{\pm 1} = K_j^{\pm 1} K_i^{\pm 1} \\ K_i K_i^{-1} = 1 \end{cases} \tag{3.1}$$

$$\begin{cases} K_i \left(E_j\right) K_i^{-1} = \epsilon^{a_{ij}} E_j \\ K_i \left(F_j\right) K_i^{-1} = \epsilon^{-a_{ij}} F_j \end{cases} \tag{3.2}$$

$$\begin{cases} [E_i, F_i] = t\delta_{ij} L_i \\ \left(ad_{\sigma_{-\alpha_i}} E_i\right)^{1-a_{ij}} E_j = 0 \\ \left(ad_{\sigma_{-\alpha_i}} F_i\right)^{1-a_{ij}} F_j = 0 \end{cases} \tag{3.3}$$

$$\begin{cases} \left(\epsilon^{d_i} - \epsilon^{-d_i}\right) L_i = t\left(K_i - K_i^{-1}\right) \\ [L_i, E_j] = t\frac{\epsilon^{a_{ij}} - 1}{\epsilon^{d_i} - \epsilon^{-d_i}} \left(E_j K_i + K_i^{-1} E_j\right) \\ [L_i, F_j] = t\frac{\epsilon^{-a_{ij}} - 1}{\epsilon^{d_i} - \epsilon^{-d_i}} \left(F_j K_i + K_i^{-1} F_j\right) \end{cases} \tag{3.4}$$

Let $0 \neq \lambda \in \mathbb{C}$, define

$$\vartheta_\lambda(E_i) = \frac{1}{\lambda} E_i,\ \vartheta_\lambda(F_i) = \frac{1}{\lambda} F_i,\ \vartheta_\lambda(L_i) = \frac{1}{\lambda} L_i,\ \vartheta_\lambda(K_i^{\pm 1}) = K_i^{\pm 1}, \tag{3.5}$$

for $i = 1, \ldots, n$.

Proposition 13. *For any $0 \neq \lambda \in \mathbb{C}$, ϑ_λ is an isomorphism of algebra between $\mathcal{U}_\epsilon^t$ and $\mathcal{U}_\epsilon^{\lambda t}$. In particular if $t \neq 0$ then $\mathcal{U}_\epsilon^t \cong \mathcal{U}_\epsilon(\mathfrak{g})$.*

Proof. Simple verification of the properties. □

Set $\mathcal{S}_\epsilon := \mathcal{U}_\epsilon^{t=0}$, we want to construct an explicit realization of it. Let $\mathcal{D} = \mathcal{U}_\epsilon(\mathfrak{b}_+) \otimes \mathcal{U}_\epsilon(\mathfrak{b}_-)$ and define the map

$$\Sigma : \mathcal{S}_\epsilon \to \mathcal{D}$$

by $\Sigma(E_i) = \mathcal{E}_i := E_i \otimes 1$, $\Sigma(F_i) = \mathcal{F}_i = 1 \otimes F_i$, and $\Sigma(K_i^{\pm 1}) = \mathcal{K}_i^{\pm 1} := K_i^{\pm 1} \otimes K_i^{\pm 1}$ for $i = 1, \ldots, n$.

Lemma 3. *Σ is a well defined map.*

Proof. We must verify that the image of E_i, F_i and K_i satisfy the relation 3.1 for $t = 0$. This is a simple matter of bookkeeping. □

Note that Σ is injective, then we can identify $\mathcal{S}_\epsilon$ with the subalgebra of $\mathcal{D}$ generated by $\mathcal{E}_i$, $\mathcal{F}_i$ and $\mathcal{K}_i$, for $i = 1, \ldots, n$. We define now the analogues of the root vectors for $\mathcal{S}_\epsilon$:

Definition 8. *For all $i = 1, \ldots, N$, let*

(i) $\mathcal{E}_{\beta_i} := E_{\beta_i} \otimes 1 \in \mathcal{S}_\epsilon$
(ii) $\mathcal{F}_{\beta_i} := 1 \otimes F_{\beta_i} \in \mathcal{S}_\epsilon$

As a consequence of this we get a PBW theorem for $\mathcal{S}_\epsilon$.

Proposition 14. *The monomials*

$$\mathcal{E}_{\beta_1}^{k_1} \ldots \mathcal{E}_{\beta_N}^{k_N} \mathcal{K}_1^{s_1} \ldots \mathcal{K}_n^{s_n} \mathcal{F}_{\beta_N}^{h_1} \ldots \mathcal{F}_{\beta_1}^{k_1}$$

for $(k_1,\dots,k_N)$, $(h_1,\dots,h_N) \in (\mathbb{Z}^+)^N$ *and* $(s_1,\dots,s_n) \in \mathbb{Z}^n$, *form a* $\mathbb{C}$ *basis of* $\mathcal{S}_\epsilon$. *Moreover*

$$\mathcal{S}_\epsilon = \mathcal{S}_\epsilon^- \otimes \mathcal{S}_\epsilon^0 \otimes \mathcal{S}_\epsilon^+$$

where $\mathcal{S}_\epsilon^+$ *(resp.* $\mathcal{S}_\epsilon^-$ *and* $\mathcal{S}_\epsilon^0$*) is the subalgebra generated by* $\mathcal{E}_{\beta_i}$ *(resp.* $\mathcal{F}_{\beta_i}$ *and* $\mathcal{K}_i$*).*

Proof. This follows from the injectivity of Σ and PBW theorem for $\mathcal{U}_\epsilon(\mathfrak{g})$ □

It is clear that $\mathcal{E}_{\beta_i}$ is also the image of the element $E_{\beta_i} \in \mathcal{U}_\epsilon^t$, where the E_{β_i} are non commutative polynomials in the E_i's by Lusztig procedure ([Lus93]). The same thing is true for $\mathcal{F}_{\beta_i}$ and F_{β_i}.

It is also clear that the LS relations hold for $\mathcal{E}_{\beta_i}$ and the $\mathcal{F}_{\beta_i}$ (instead of the E_{β_i} and the F_{β_i}) inside $\mathcal{S}_\epsilon$.

Theorem 4. $\mathcal{S}_\epsilon = \mathcal{U}_\epsilon^{t=0}$ *is a twisted derivation algebra.*

Proof. Define $\mathcal{U}^0 = \mathbb{C}[\mathcal{E}_{\beta_1}, \mathcal{F}_{\beta_N}] \subset \mathcal{S}_\epsilon$, then we can define

$$\mathcal{U}^i = \mathcal{U}^{i-1}_{\sigma,D}\left[\mathcal{E}_{\beta_i}, \mathcal{F}_{\beta_{N-i}}\right] \subset \mathcal{S}_\epsilon$$

where σ and D are given by the L.S. relation. Note now that, the $\mathcal{K}_i$, for $i = 1,\dots,n$ normalize $\mathcal{U}^N$, and when we add them to this algebra we perform an iterated construction of twisted Laurent polynomial. The resulting algebra will be called $\mathcal{T}$. We now claim

$$\mathcal{S}_\epsilon = \mathcal{T}$$

Note that, by construction $\mathcal{T} \subset \mathcal{S}_\epsilon$, so we only have to prove that $\mathcal{S}_\epsilon \subset \mathcal{T}$. Now note that

$$\mathcal{E}_{\beta_1}^{k_1} \cdots \mathcal{E}_{\beta_N}^{k_N} \mathcal{K}_1^{s_1} \cdots \mathcal{K}_n^{s_n} \mathcal{F}_{\beta_N}^{h_N} \cdots \mathcal{F}_{\beta_1}^{h_1} \in \mathcal{T}$$

for every $(k_1,\dots,k_N)$, $(h_1,\dots,h_N) \in (\mathbb{Z}^+)^N$ and $(s_1,\dots,s_n) \in \mathbb{Z}^n$. Then by proposition 14 we have $\mathcal{S}_\epsilon \subset \mathcal{T}$. □

We finish this section with some remarks on the center of $\mathcal{U}_\epsilon^t$. Recall that $\mathcal{U}_\epsilon^t$ is isomorphic to $\mathcal{U}_\epsilon$ for every $t \in \mathbb{C}^*$, hence Z_ϵ^t is isomorphic to $Z_\epsilon^1 = Z_\epsilon$. For $t = 0$, we define C_0 the subalgebra of $\mathcal{S}_\epsilon$ generated by $\mathcal{E}_\beta^l$, $\mathcal{F}_\beta^l$ for $\beta \in R^+$ and $\mathcal{K}_j^{\pm l}$ for $j = 1,\dots,n$ and let C_ϵ be the center of $\mathcal{S}_\epsilon$. Let $Z_0[t]$ the trivial deformation of Z_0

Lemma 4. *(i)* $\rho : Z_0[t] \to \mathcal{U}_\epsilon^t$ *defined in the obvious way is an injective homomorphism of algebra.*

(ii) $\mathcal{U}_\epsilon^t$ *is a free* $Z_0[t]$ *module with base the set of monomials*

$$E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N} K_1^{s_1} \cdots K_n^{s_n} F_{\beta_N}^{h_N} \cdots F_{\beta_1}^{h_1}$$

for which $0 \le k_i, s_j, h_i < l$, *for* $i = 1,\dots,N$ *and* $j = 1,\dots,n$.

Proof. (i) follows by definitions of $Z_0[t]$. (ii) follows from the PBW theorem. □

Lemma 5. *(i)* $Z_0 \cong C_0$.

(ii) $\mathcal{U}_\epsilon$ *and* $\mathcal{S}_\epsilon$ *are isomorphic has* Z_0 *modules.*

Proof. Follows from the definitions. □

3.2. **General case.** We can now study the general case.

Definition 9. *Let $\mathcal{U}_\epsilon^t(\mathfrak{p})$ be the subalgebra of $\mathcal{U}_\epsilon^t$ generated by $E_{\beta_i^1}$, F_{β_j} and $K_s^{\pm 1}$ for $i = 1, \dots, h$, $j = 1, \dots, N$ and $s = 1, \dots, n$.*

Set $\mathcal{S}_\epsilon(\mathfrak{p}) = \mathcal{U}_\epsilon^{t=0}(\mathfrak{p}) \subset \mathcal{S}_\epsilon$.

Proposition 15. *(i) For every $t \in \mathbb{C}$, $\mathcal{U}_\epsilon^t(\mathfrak{p})$ is a Hopf subalgebra of $\mathcal{U}_\epsilon^t$.*
(ii) For any $\lambda \neq 0$, ϑ_λ defines by 3.5 is an algebra isomorphism between $\mathcal{U}_\epsilon^t(\mathfrak{p})$ and $\mathcal{U}_\epsilon^{\lambda t}(\mathfrak{p})$.

Proof. This is an immediate consequence of the same properties in the case $\mathfrak{p} = \mathfrak{g}$. □

We can now state the main theorem of this section

Theorem 5. *$\mathcal{S}_\epsilon(\mathfrak{p})$ is a twisted derivation algebra*

Proof. We use the same technique as we used in the proof of theorem 4. Let $\mathcal{D}(\mathfrak{p}) = \mathcal{U}_\epsilon(\mathfrak{b}_+^{\mathfrak{l}}) \otimes \mathcal{U}_\epsilon(\mathfrak{b}_-)$. Define

$$\Sigma : \mathcal{S}_\epsilon(\mathfrak{p}) \to \mathcal{D}(\mathfrak{p})$$

by $\Sigma(E_i) = \mathcal{E}_i$, $\Sigma(F_j) = \mathcal{F}_j$, $\Sigma(K_j^{\pm 1}) = \mathcal{K}_j^{\pm 1}$ for $i \in \Pi^{\mathfrak{l}}$ and $j = 1, \dots, n$.

Lemma 6. *$\mathcal{S}_\epsilon(\mathfrak{p})$ is a subalgebra of $\mathcal{S}_\epsilon$*

Proof. Note that $\mathcal{D}(\mathfrak{p})$ is a subalgebra of $\mathcal{D}$, and, as in lemma 3, the map Σ is well defined and injective. So, we have the following commutative diagram

$$\begin{tikzcd} \mathcal{S}_\epsilon(\mathfrak{p}) \arrow[r, "\Sigma"] \arrow[d, "i"'] & \mathcal{D}(\mathfrak{p}) \arrow[d, "j"] \\ \mathcal{S}_\epsilon \arrow[r, "\Sigma"] & \mathcal{D} \end{tikzcd}$$

Since Σ and j are injective maps, we have that i is also injective □

So we can identify $\mathcal{S}_\epsilon(\mathfrak{p})$ with the subalgebra of $\mathcal{S}_\epsilon$ generated by $\mathcal{E}_{\beta_i^1}$, $\mathcal{F}_{\beta_s}$ and $\mathcal{K}_j^{\pm 1}$ for $i = 1, \dots, h$, $s = 1, \dots, N$ and $j = 1, \dots, n$. As a corollary of proposition 14 and LS relations, we have:

Proposition 16. *(i) The monomials*

$$\mathcal{E}_{\beta_1^1}^{k_1} \dots \mathcal{E}_{\beta_h^1}^{k_h} \mathcal{K}_1^{s_1} \dots \mathcal{K}_n^{s_n} \mathcal{F}_{\beta_N}^{t_1} \dots \mathcal{F}_{\beta_1}^{t_1}$$

for $(k_1, \dots, k_h) \in (\mathbb{Z}^+)^h$, $(t_1, \dots, t_N) \in (\mathbb{Z}^+)^N$ and $(s_1, \dots, s_n) \in \mathbb{Z}^n$, form a $\mathbb{C}$ basis of $\mathcal{S}_\epsilon(\mathfrak{p})$.
(ii) For $i < j$ one has
(a)

$$\mathcal{E}_{\beta_j^1}\mathcal{E}_{\beta_i^1} - \epsilon^{(\beta_i^1|\beta_j^1)}\mathcal{E}_{\beta_i^1}\mathcal{E}_{\beta_j^1} = \sum_{k \in \mathbb{Z}_+^N} c_k \mathcal{E}^k \tag{3.6}$$

where $c_k \in \mathbb{C}$ and $c_k \neq 0$ only when $k = (k_1, \dots, k_h)$ is such that $k_s = 0$ for $s \leq i$ and $s \geq j$, and $\mathcal{E}^k = \mathcal{E}_{\beta_1^1}^{k_1} \dots \mathcal{E}_{\beta_h^1}^{k_h}$.

(b)

$$\mathcal{F}_{\beta_j}\mathcal{F}_{\beta_i} - \epsilon^{-(\beta_i|\beta_j)}\mathcal{F}_{\beta_i}\mathcal{F}_{\beta_j} = \sum_{k\in\mathbb{Z}_+^N} c_k \mathcal{F}^k \tag{3.7}$$

where $c_k \in \mathbb{C}$ *and* $c_k \neq 0$ *only when* $k = (k_1, \ldots, k_N)$ *is such that* $k_s = 0$ *for* $s \leq i$ *and* $s \geq j$, *and* $\mathcal{F}^k = \mathcal{F}_{\beta_N}^{k_N} \ldots \mathcal{F}_{\beta_1}^{k_1}$.

So we have:

Theorem 6. *The monomials*

$$\mathcal{E}_{\beta_1^1}^{k_1} \ldots \mathcal{E}_{\beta_h^1}^{k_h} \mathcal{K}_1^{s_1} \ldots \mathcal{K}_n^{s_n} \mathcal{F}_{\beta_N}^{t_N} \ldots \mathcal{F}_{\beta_1}^{t_1}$$

for $(k_1, \ldots, k_h) \in (\mathbb{Z}^+)^h$, $(t_1, \ldots, t_N) \in (\mathbb{Z}^+)^N$ *and* $(s_1, \ldots, s_n) \in \mathbb{Z}^n$, *are a* $\mathbb{C}[t]$ *basis of* $\mathcal{U}_\epsilon^t$. *In particular* t *is not a zero divisor in* $\mathcal{U}_\epsilon^t$ *hence* $\mathcal{U}_\epsilon^t$ *is a flat over* $\mathbb{C}[t]$

As we see in general case, we can conclude that $\mathcal{S}_\epsilon(\mathfrak{p})$ is a quasi polynomial algebra. □

4. The degree

4.1. **The degree of $\mathcal{S}_\epsilon(\mathfrak{p})$.** Using the method exposed in [DCP93] we can now start the calculation of the degree of $\mathcal{U}_\epsilon(\mathfrak{p})$.

Theorem 7. *If* l *is a good integer, then*

$$\deg \mathcal{S}_\epsilon(\mathfrak{p}) = l^{\frac{1}{2}\left(l(w_0)+l\left(w_0^{\mathfrak{l}}\right)+\operatorname{rank}\left(w_0-w_0^{\mathfrak{l}}\right)\right)}$$

Proof. Denote by $\overline{\mathcal{S}}_\epsilon(\mathfrak{p})$ the quasi polynomial algebra associated to $\mathcal{S}_\epsilon(\mathfrak{p})$. We know by the general theory that

$$\deg \mathcal{S}_\epsilon(\mathfrak{p}) = \deg \overline{\mathcal{S}}_\epsilon(\mathfrak{p}).$$

Let x_i denote the class of $E_{\beta_i^1}$ in $\overline{\mathcal{S}}_\epsilon(\mathfrak{p})$ for $i = 1, \ldots, h$ and y_j the class of F_{β_j} for $j = 1, \ldots, N$, then from theorem 5 we have

$$x_i x_j = \epsilon^{(\beta_i^1|\beta_j^1)} x_j x_i, \tag{4.1}$$

$$y_i y_j = \epsilon^{-(\beta_i|\beta_j)} y_j y_i. \tag{4.2}$$

if $i < j$. Thus we introduce the skew symmetric matrices $A = (a_{ij})$ with $a_{ij} = (\beta_i|\beta_j)$ for $i < j$ and $A^{\mathfrak{l}} = \left(a'_{ij}\right)$ with $a'_{ij} = (\beta_i^1|\beta_j^1)$ for $i < j$.

Let k_i be the class of K_i, using the relation in theorem 5 we obtain a $n \times N$ matrix $B = ((w_i|\beta_j))$ and a $h \times N$ matrix $B^{\mathfrak{l}} = ((w_i|\beta_j^1))$.

Let $t = 2$ unless the Cartan matrix is of type G_2, in which case $t = 6$. Since we will eventually reduce modulo l an odd integer coprime with t, we start inverting t. Thus consider the free $\mathbb{Z}\left[\frac{1}{t}\right]$ module V^+ with basis $u_1, \ldots, u_h$, V^- with basis $u'_1, \ldots, u'_N$ and V^0 with basis $w_1, \ldots, w_n$. On $V = V^+ \oplus V^0 \oplus V^-$ consider the bilinear form given by

$$T = \begin{pmatrix} A^{\mathfrak{l}} & -{}^tB^{\mathfrak{l}} & 0 \\ B^{\mathfrak{l}} & 0 & -B \\ 0 & {}^tB & -A \end{pmatrix},$$

then the rank of T is the degree of $\overline{\mathcal{S}}_\epsilon(\mathfrak{p})$.

Consider the operators $M^{\mathfrak{l}} = \begin{pmatrix} A^{\mathfrak{l}} & -{}^tB^{\mathfrak{l}} & 0 \end{pmatrix}$, $M = \begin{pmatrix} 0 & {}^tB & -A \end{pmatrix}$, and $N = \begin{pmatrix} B^{\mathfrak{l}} & 0 & -B \end{pmatrix}$, so that $T = M^{\mathfrak{l}} \oplus N \oplus M$.

Note that
$$B(u_i') = \beta_i$$
and
$$B^{\mathfrak{l}}(u_i) = \beta_i^1$$
Now we need some technical lemma:

Lemma 7. *Let $w \in \mathcal{W}$ and fix a reduced expression $w = s_{i_1} \cdots s_{i_k}$. Given $\omega = \sum_{i=1}^n \delta_i \mathbf{w}_i$, with $\delta_i = 0$ or 1. Set*
$$I_\omega(w) := \{t \in \{1, \ldots, k\} : s_{i_t}(\omega) \neq \omega\}.$$
Then
$$\omega - w(\omega) = \sum_{t \in I_\omega} \beta_t$$

Proof. We proceed by induction on the length of w. The hypothesis made implies $s_i(\omega) = \omega$ or $s_i(\omega) = \omega - \alpha_i$. Write $w = w' s_{i_k}$. If $k \notin I_\omega$, then $w(\omega) = w'(\omega)$ and we are done by induction. Otherwise
$$w(\omega) = w'(\omega - \alpha_{i_k}) = w'(\omega) - \beta_k$$
and again we are done by induction. □

Lemma 8. *Let $\theta = \sum_{i=1}^n a_i \alpha_i$ the highest root of the root system R. Let $\mathbb{Z}'' = \mathbb{Z}[a_1^{-1}, \ldots, a_n^{-1}]$, and let $\Lambda'' = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}''$ and $Q'' = Q \otimes_{\mathbb{Z}} \mathbb{Z}''$. Then the $\mathbb{Z}''$ submodule $(1 - w_0)\Lambda''$ of Q'' is a direct summand.*

Proof. See [DCKP95] or [DCP93] §10. □

Lemma 9. *(i) The operator M is surjective*

(ii) The vector $v_\omega := \left(\sum_{t \in I_\omega} u_t\right) - \omega - w_0(\omega)$, as ω run thought the fundamental weights, form a basis of the kernel of M.

(iii) $N(v_\omega) = \omega - w_0(\omega) = \sum_{t \in I_\omega} \beta_t$.

Proof. See [DCKP95] or [DCP93] §10. □

Set $T_1 = M^{\mathfrak{l}} \oplus M$, then using the notation of lemma 7, we have

Lemma 10. *The vector $v_{\mathbf{w}} = \sum_{t \in I_{\mathbf{w}}(w_0^{\mathfrak{l}})} u_t - \mathbf{w} - w_0(\mathbf{w}) + \sum_{t \in I_{\mathbf{w}}(w_0)} u_t'$, as $\mathbf{w}$ runs through the fundamental weights, form a basis of the kernel of T_1.*

Proof. First, we observe that T_1 is onto, since M and $M^{\mathfrak{l}}$ are projections over V^- and V^+ respectively, by lemma 9. Since the n vectors $v_{\mathbf{w}}$ are part of a basis and, the kernel of T_1 is a direct summand of rank n, by surjectivity. It is enough to show that $v_{\mathbf{w}}$ is in the kernel of T_1. We have
$$\begin{aligned} T_1(v_{\mathbf{w}}) &= A^{\mathfrak{l}} \left(\sum_{t \in I_{\mathbf{w}}(w_0^{\mathfrak{l}})} u_t \right) -^t B^{\mathfrak{l}} \left(-\mathbf{w} - w_0(\mathbf{w})\right) \\ &\quad +^t B \left(-\mathbf{w} - w_0(\mathbf{w})\right) - A \left(\sum_{t \in I_{\mathbf{w}}(w_0)} u_t' \right) \\ &= M^{\mathfrak{l}}(v_{\mathbf{w}}) -^t B^{\mathfrak{l}} \left(w_0^{\mathfrak{l}}(\mathbf{w}) - w_0(\mathbf{w})\right) - M(v_{\mathbf{w}}). \end{aligned}$$
So from lemma 7 and lemma 9, we have:
$$T_1(v_w) = -^t B^{\mathfrak{l}} \left(w_0^{\mathfrak{l}}(\mathbf{w}) - w_0(\mathbf{w})\right)$$

Let $w_0 = w_0^{\mathfrak{l}}\overline{w}$, since w runs through the fundamental weights, we have two cases:

(1) $\overline{w}(\mathbf{w}) = w$, therefore $w_0^{\mathfrak{l}}(\mathbf{w}) - w_0(\mathbf{w}) = 0$ and $T_1(v_{\mathbf{w}}) = 0$.

(2) $\overline{w}(\mathbf{w}) \neq \mathbf{w}$, therefore $w_0^{\mathfrak{l}}(\mathbf{w}) = \mathbf{w}$ and $w_0^{\mathfrak{l}}(\mathbf{w}) - w_0(\mathbf{w}) = \mathbf{w} - w_0(\mathbf{w}) \in \ker {}^tB^{\mathfrak{l}}$, by definition of ${}^tB^{\mathfrak{l}}$, so $T_1(v_{\mathbf{w}}) = 0$.

□

Since T is the direct sum of T_1 and N, its kernel is the intersection of the 2 kernels of these operators. We have computed the kernel of T_1 in lemma 10. Thus the kernel of T equals the kernel of N restricted to the submodule spanned by the $v_{\mathbf{w}}$.

Lemma 11.

$$N(v_{\mathbf{w}}) = \sum_{t \in I_{\mathbf{w}}(w_0^{\mathfrak{l}})} \beta_t^1 - \sum_{t \in I_{\mathbf{w}}(w_0)} \beta_t = w_0(\mathbf{w}) - w_0^{\mathfrak{l}}(\mathbf{w}).$$

Proof. Note that $B(u_t) = \beta_t$, then

$$N(v_w) = \sum_{t \in I_{\mathbf{w}}(w_0^{\mathfrak{l}})} \beta_t^1 - \sum_{t \in I_{\mathbf{w}}(w_0)} \beta_t. \tag{4.3}$$

Finally, the claim follows using lemma 7. □

Thus, we can identify N we the map $w_0 - w_0^{\mathfrak{l}} : \Lambda \to Q$. At this point we need the following fact

Lemma 12. *Let $\theta = \sum_{i=1}^n a_i\alpha_i$ the highest root of the root system R. Let $\mathbb{Z}' = \mathbb{Z}[a_1^{-1}, \ldots, a_n^{-1}]$, and let $\Lambda' = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}'$ and $Q' = Q \otimes_{\mathbb{Z}} \mathbb{Z}'$. Then the $\mathbb{Z}'$ submodule $(w_0 - w_0^{\mathfrak{l}})\Lambda'$ of Q' is a direct summand.*

Proof. The claim follows as a consequence of lemma 8. □

So if l is a good integer, i.e. l is coprime with t and a_i for all i, we have

$$\operatorname{rank} T = l(w_0) + l(w_0^{\mathfrak{l}}) + n - \left(n - \operatorname{rank}\left(w_0 - w_0^{\mathfrak{l}}\right)\right),$$

and so the theorem follows. □

4.2. **The degree of $\mathcal{U}_\epsilon(\mathfrak{p})$.** We begin by observing that every irreducible $\mathcal{U}_\epsilon(\mathfrak{p})$ module V is finite dimensional. Indeed, let $\mathcal{Z}(V)$ be the subalgebra of the algebra of intertwining operators of V generated by the action of the elements in $Z_\epsilon(\mathfrak{p})$. Since $\mathcal{U}_\epsilon(\mathfrak{p})$ is finitely generated as a $Z_\epsilon(\mathfrak{p})$ module, V is finitely generated as $\mathcal{Z}(V)$ module. If $0 \neq f \in \mathcal{Z}(V)$, then $f \cdot V = V$, otherwise $f \cdot V$ is a proper submodule V. Hence, by Nakayama's lemma, there exist an endomorphism $g \in \mathcal{Z}(V)$ such that $1 - gf = 0$, i.e. f is invertible. Thus $\mathcal{Z}(V)$ is a field. It follows easily that $\mathcal{Z}(V)$ consists of scalar operators. Thus V is a finite dimensional vector space.

Since $Z_\epsilon(\mathfrak{p})$ acts by scalar operators on V, there exists an homomorphism $\chi_V : Z_\epsilon \mapsto \mathbb{C}$, the *central character of* V, such that

$$z \cdot v = \chi_V(z)v$$

for all $z \in Z_\epsilon$ and $v \in V$. Note that isomorphic representations have the same central character, so assigning to a $\mathcal{U}_\epsilon(\mathfrak{p})$ module its central character gives a well defined map

$$\Xi : \operatorname{Rap}\left(\mathcal{U}_\epsilon(\mathfrak{p})\right) \to \operatorname{Spec}\left(Z_\epsilon(\mathfrak{p})\right),$$

where $\mathrm{Rap}\,(\mathcal{U}_\epsilon(\mathfrak{p}))$ is the set of isomorphism classes of irreducible $\mathcal{U}_\epsilon(\mathfrak{p})$ modules, and $\mathrm{Spec}\,(Z_\epsilon(\mathfrak{p}))$ is the set of algebraic homomorphisms $Z_\epsilon(\mathfrak{p}) \mapsto \mathbb{C}$.

To see that Ξ is surjective, let I^χ, for $\chi \in \mathrm{Spec}\,(Z_\epsilon(\mathfrak{p}))$, be the ideal in $\mathcal{U}_\epsilon(\mathfrak{p})$ generated by

$$\ker\chi = \{z - \chi(z)\cdot 1 : z \in Z_\epsilon(\mathfrak{p})\}\,.$$

To construct $V \in \Xi^{-1}(\chi)$ is the same as to construct an irreducible representation of the algebra $\mathcal{U}_\epsilon^\chi(\mathfrak{p}) = \mathcal{U}_\epsilon(\mathfrak{p})/I^\chi$. Note that $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ is finite dimensional and non zero. Thus, we may take V, for example, to be any irreducible subrepresentation of the regular representation of $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$.

Let $\chi \in \mathrm{Spec}(Z_0(\mathfrak{p}))$, we define,

$$\mathcal{U}_\epsilon^\chi(\mathfrak{p}) = \mathcal{U}_\epsilon(\mathfrak{p})/J^\chi$$

where J^χ is the two sided ideal generated by

$$\ker\chi = \{z - \chi(z)\cdot 1 : z \in Z_0(\mathfrak{p})\}$$

As we have seen at the end of section 3.1, $Z_0(\mathfrak{p})[t] \subset \mathcal{U}_\epsilon^t(\mathfrak{p})$, so for all $t \in \mathbb{C}$ and $\chi \in \mathrm{Spec}(Z_0(\mathfrak{p}))$, we can define $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p}) = \mathcal{U}_\epsilon^t(\mathfrak{p})/J^\chi$ where J^χ is the two side ideal generated by

$$\ker\chi = \{z - \chi(z)\cdot 1 : z \in Z_0(\mathfrak{p})\}$$

The PBW theorem for $\mathcal{U}_\epsilon^t(\mathfrak{p})$ implies that

Proposition 17. *The monomials*

$$E_{\beta_1^1}^{s_1}\cdots E_{\beta_h^1}^{s_h} K_1^{r_1}\cdots K_n^{r_n} F_{\beta_h^1}^{t_{k+h}}\cdots F_{\beta_1^1}^{t_{k+1}} F_{\beta_k^2}^{t_k}\cdots F_{\beta_1^2}^{t_1}$$

for which $0 \le s_j, t_i, r_v < l$, *form a* $\mathbb{C}$ *basis for* $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$

Lemma 13. *For every* $0 \ne \lambda \in \mathbb{C}$, $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$ *is isomorphic to* $\mathcal{U}_\epsilon^{\lambda t,\chi}(\mathfrak{p})$.

Proof. Consider the isomorphism ϑ_λ from $\mathcal{U}_\epsilon^t(\mathfrak{p})$ and $\mathcal{U}_\epsilon^{\lambda t}(\mathfrak{p})$, defined by (3.5). Its follows from the above definition that $\vartheta_\lambda(J^\chi) = J^\chi$. Then ϑ_λ induce an isomorphism between $\mathcal{U}_\epsilon^{t,\chi}$ and $\mathcal{U}_\epsilon^{\lambda t,\chi}$. □

Proposition 18. *The* $\mathcal{U}_\epsilon(\mathfrak{p})$ *algebras* $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$ *form a continuous family parametrized by* $\mathcal{Z} = \mathbb{C} \times \mathrm{Spec}(Z_0(\mathfrak{p}))$.

Proof. Let $\mathcal{V}$ denote the set of triple (t,χ,u) with $(t,\chi) \in \mathcal{Z}$ and $u \in \mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$. Then from the PBW theorem we have that the set of monomial

$$E_{\beta_1}^{s_1}\cdots E_{\beta_h^1}^{s_h} K_1^{r_1}\ldots K_n^{r_n} F_{\beta_N}^{t_N}\cdots F_{\beta_1}^{t_1}$$

for which $0 \le s_i, t_i, r_v < l$, for $i \in \Pi^{\mathfrak{l}}$, $j = 1, \ldots, N$ and $v = \ldots, n$, form a basis for each algebra $\mathcal{U}_\epsilon^{t,\chi}$.

Order the previous monomials and assign to $u \in \mathcal{U}_\epsilon^{t,\chi}$ the coordinate vector of u with respect to the ordered basis. This construction identifies $\mathcal{V}$ with $\mathcal{Z} \times \mathbb{C}^d$, where $d = l^{h+n+N}$, thereby giving $\mathcal{A}$ a structure of an affine variety.

Consider the vector bundle $\pi : \mathcal{V} \to \mathcal{Z}$, $(t,\chi,u) \to (t,\chi)$. Note that the structure constant of the algebra $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$, as well as the matrix entries of the linear transformations which define the action of $\mathcal{U}_\epsilon(\mathfrak{p})$ relative to the basis, are polynomial in χ and t. This means that the maps

$$\mu : \mathcal{V} \times_{\mathcal{Z}} \mathcal{V} \to \mathcal{V}, \qquad ((t,\chi,u),(t,\chi,v)) \mapsto (t,\chi,uv)$$
$$\rho : \mathcal{U}_\epsilon \times \mathcal{V} \to \mathcal{V}, \qquad (x,(t,\chi,u)) \mapsto (t,\chi,x\cdot u)$$

where $(t,\chi) \in \mathcal{Z}$, u, $v \in \mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$ and $x \in \mathcal{U}_\epsilon(\mathfrak{p})$, define on $\mathcal{V}$ a structure of vector bundle of algebra and a structure of vector bundle of $\mathcal{U}_\epsilon(\mathfrak{p})$-modules. The fiber of π above (t,χ) is the $\mathcal{U}_\epsilon(\mathfrak{p})$-algebra $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$. □

If we fix $\chi \in \operatorname{Spec}(Z_0)$, we have from theorem 6 that the family of algebra $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$ is a flat deformation of algebra over $\operatorname{Spec}\mathbb{C}[t]$.

Summarizing, if ϵ is a primitive l^{th} root of 1 with l odd and $l > d_i$ for all i, we have prove the following facts on $\mathcal{U}_\epsilon$:

- $\mathcal{U}_\epsilon^t$ and $U_\epsilon(\mathfrak{p})$ are domains because $\mathcal{U}_\epsilon(\mathfrak{g})$ it is,
- $\mathcal{U}_\epsilon^t$ and $\mathcal{U}_\epsilon(\mathfrak{p})$ are finite modules over $Z_0[t]$ and Z_0 respectively (cf lemma 4 and proposition 10).

Since the L.S. relations holds for $\mathcal{U}_\epsilon(\mathfrak{p})$ and $\mathcal{U}_\epsilon^t$ (cf proposition 8), we can apply the theory developed in [DCP93], and we obtain that $\operatorname{Gr}\mathcal{U}_\epsilon(\mathfrak{p})$ and $\operatorname{Gr}\mathcal{U}_\epsilon^t$ are twisted polynomial algebra, with some elements inverted. Hence all conditions of the characterization of maximal order (see theorem 6.4 in [DCP93]) are verified, so

Theorem 8. *U_ϵ^t and $\mathcal{U}_\epsilon(\mathfrak{p})$ are maximal orders.*

Therefore, $\mathcal{U}_\epsilon(\mathfrak{p}) \in \mathcal{C}_m$, i.e. is an algebra with trace of degree m.

Theorem 9. *The set*

$$\Omega = \{a \in \operatorname{Spec}(Z_\epsilon(\mathfrak{p})), \text{ such that the corresponding semisimple representation of } \mathcal{U}_\epsilon(\mathfrak{p}) \text{ is irreducible}\}$$

is a Zariski open set. This is exactly the part of $\operatorname{Spec}(Z_\epsilon(\mathfrak{p}))$ *over which* $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ *is an Azumaya algebra of degree m.*

Proof. Apply theorem 4.5 in [DCP93], with $R = \mathcal{U}_\epsilon(\mathfrak{p})$ and $T = Z_\epsilon(\mathfrak{p})$. □

Recall that $Z_\epsilon(\mathfrak{p})$ is a finitely generated module over $Z_0(\mathfrak{p})$. Let $\tau : \operatorname{Spec}(Z_\epsilon(\mathfrak{p})) \to \operatorname{Spec}(Z_0(\mathfrak{p}))$ be the finite surjective morphism induced by the inclusion of $Z_0(\mathfrak{p})$ in $Z_\epsilon(\mathfrak{p})$. The properness of τ implies the following

Corollary 1. *The set*

$$\Omega_0 = \left\{a \in \operatorname{Spec}(Z_0(\mathfrak{p})) : \tau^{-1}(a) \subset \Omega\right\}$$

is a Zariski dense open subset of $\operatorname{Spec}(Z_0)$.

We know by the theory developed in [DCP93], that $\mathcal{S}_\epsilon(\mathfrak{p}) \in \mathcal{C}_{m_0}$, with $m_0 = l^{l(w_0)+l(w_0^{\mathfrak{l}})+\operatorname{rank}\left(w_0-w_0^{\mathfrak{l}}\right)}$. As we see in proposition 4, $S_\epsilon(\mathfrak{p})$ is a finite module over C_0, then C_ϵ, the center of $\mathcal{S}_\epsilon(\mathfrak{p})$ is finite over C_0. The inclusion $C_0 \hookrightarrow C_\epsilon$ induces a projection $\upsilon : \operatorname{Spec}(C_\epsilon) \to \operatorname{Spec}(C_0)$. As before, we have:

Lemma 14. *(i)*

$$\Omega' = \{a \in \operatorname{Spec}(C_\epsilon), \text{such that the corresponding semisimple representation of } \mathcal{S}_\epsilon(\mathfrak{p}) \text{ is irreducible}\}$$

is a Zariski open set. This is exactly the part of $\operatorname{Spec}(C_\epsilon)$ *over which* $\mathcal{S}_\epsilon^\chi(\mathfrak{p})$ *is an Azumaya algebra of degree m_0.*

(ii) The set

$$\Omega_0' = \left\{a \in \operatorname{Spec}(Z_0(\mathfrak{p})) : \upsilon^{-1}(a) \subset \Omega'\right\}$$

is a Zariski dense open subset of $\operatorname{Spec}(Z_0)$.

Proof. Apply theorem 4.5 in [DCP93] at $\mathcal{S}_\epsilon(\mathfrak{p})$. $\square$

Since $\mathrm{Spec}(Z_0)$ is irreducible, we have that $\Omega_0 \cap \Omega_0'$ is non empty.
We can state the main theorem of this section

Theorem 10.
$$\deg \mathcal{U}_\epsilon(\mathfrak{p}) = l^{\frac{1}{2}\left(l(w_0)+l(w_0^{\mathfrak{l}})+\mathrm{rank}\left(w_0-w_0^{\mathfrak{l}}\right)\right)}$$

Proof. For $\chi \in \Omega_0 \cap \Omega_0'$, we have, using theorem 9 and lemma 14,
$$\deg \mathcal{U}_\epsilon(\mathfrak{p}) = m = \deg \mathcal{U}_\epsilon^\chi(\mathfrak{p}),$$
and
$$\deg \mathcal{S}_\epsilon^\chi(\mathfrak{p}) = \deg \mathcal{S}_\epsilon(\mathfrak{p}).$$
But for all $t \neq 0$, we have that $\mathcal{U}_\epsilon^{t,\chi}$ is isomorphic to $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ as algebra. By construction $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ as a module over itself is irreducible, hence it is a simple algebra. By rigidity of semisimple algebra ([Pie82] or [Pro98]) we have that $\mathcal{S}_\epsilon^\chi(\mathfrak{p}) = \mathcal{U}_\epsilon^{0,\chi}$ is isomorphic to $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$. Then
$$\deg \mathcal{U}_\epsilon(\mathfrak{p}) = m = \deg \mathcal{U}_\epsilon^\chi(\mathfrak{p}) = \deg \mathcal{S}_\epsilon^\chi(\mathfrak{p}) = \deg \mathcal{S}_\epsilon(\mathfrak{p}).$$
And by theorem 7 the claim follows. $\square$

As $\mathcal{U}_\epsilon(\mathfrak{p})$ is a maximal order, $Z_\epsilon(\mathfrak{p})$ is integrally closed, so we can make the following construction: denote by $Q_\epsilon := Q(Z_\epsilon(\mathfrak{p}))$ the field of fractions of $Z_\epsilon(\mathfrak{p})$, we have that $Q(\mathcal{U}_\epsilon(\mathfrak{p})) = \mathcal{U}_\epsilon(\mathfrak{p}) \otimes_{Z_\epsilon(\mathfrak{p})} Q_\epsilon$ is a division algebra, finite dimensional over its center Q_ϵ. Denote by $\mathcal{F}$ the maximal commutative subfield of $Q(\mathcal{U}_\epsilon(\mathfrak{p}))$, we have, using standard tools of associative algebra (cf [Pie82]), that

(i) $\mathcal{F}$ is a finite extension of Q_ϵ of degree m,
(ii) $Q(\mathcal{U}_\epsilon(\mathfrak{p}))$ has dimension m^2 over Q_ϵ,
(iii) $Q(\mathcal{U}_\epsilon(\mathfrak{p})) \otimes_{Q_\epsilon} \mathcal{F} \cong M_m(\mathcal{F})$.

Hence, we have that
$$\begin{aligned}
\dim_{Q(Z_0(\mathfrak{p}))}(Q_\epsilon) &= \deg(\tau) \\
\dim_{Q_\epsilon}(Q(\mathcal{U}_\epsilon(\mathfrak{p}))) &= m^2 \\
\dim_{Q(Z_0(\mathfrak{p}))}(Q(\mathcal{U}_\epsilon(\mathfrak{p}))) &= l^{h+N+n}
\end{aligned}$$
where, the first equality is a definition, the second has been pointed out above and the third follows from the P.B.W theorem. Then, we have
$$l^{h+N+n} = m^2 \deg(\tau)$$
with $m = l^{\frac{1}{2}\left(l(w_0)+l(w_0^{\mathfrak{l}})+\mathrm{rank}\left(w_0-w_0^{\mathfrak{l}}\right)\right)}$, so

Corollary 2.
$$\deg(\tau) = l^{n-\mathrm{rank}\left(w_0-w_0^{\mathfrak{l}}\right)}.$$

5. The center

We want to make some observations on the center of $\mathcal{U}_\epsilon^{\chi,t}$ and $\mathcal{U}_\epsilon^t$ that perhaps they can be useful in the explicit determination of the center of $\mathcal{U}_\epsilon(\mathfrak{g})$.

5.1. **The center of $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$.** We want to explain a method which in principle allows us to determine the center of $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ for all $\chi \in \mathrm{Spec}(Z_0)$.

Let $\chi_0 \in \mathrm{Spec}(Z_0(\mathfrak{p}))$ define by $\chi_0(E_i) = 0$, $\chi_0(F_i) = 0$ and $\chi_0(K_i^{\pm 1}) = 1$, we set $\mathcal{U}_\epsilon^0(\mathfrak{p}) = \mathcal{U}_\epsilon^{\chi_0}(\mathfrak{p})$.

Proposition 19. *$\mathcal{U}_\epsilon^0(\mathfrak{p})$ is a Hopf algebra with the comultiplication, counit and antipode induced by $\mathcal{U}_\epsilon(\mathfrak{p})$.*

Proof. This is immediately since J^{χ_0} is an Hopf ideal. □

Proposition 20. *let $\chi \in \mathrm{Spec}(Z_0(\mathfrak{p}))$*

(i) $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ is an $\mathcal{U}_\epsilon(\mathfrak{p})$ module, with the action define by

$$a \cdot u = \sum a_{(1)} u S(a_{(2)}),$$

where $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

(ii) $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ is an $\mathcal{U}_\epsilon^0(\mathfrak{p})$ module, with the action induced by $\mathcal{U}_\epsilon(\mathfrak{p})$.

Proof. Easy verification of the proprieties. □

Proposition 21. *Let $x \in \mathcal{U}_\epsilon^\chi(\mathfrak{p})$. Then x is in the center of $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ if and only if x is invariant under the action of $\mathcal{U}_\epsilon^0(\mathfrak{p})$, that is*

$$E_i \cdot x = 0, \tag{5.1a}$$
$$F_i \cdot x = 0, \tag{5.1b}$$
$$K_i \cdot x = x. \tag{5.1c}$$

Proof. Let $x \in Z\left(\mathcal{U}_\epsilon^\chi(\mathfrak{p})\right)$ then

$$E_i \cdot x = E_i x - K_i x K_i^{-1} E_i = 0,$$

in the same way we obtain the other relations.

Suppose now that x verify the relations 5.1. Then

$$K_i \cdot x = K_i x K_i^{-1} = x,$$

imply that $K_i x = x K_i$. From $E_i \cdot x = 0$ we obtain

$$\begin{aligned} 0 = E_i \cdot x &= E_i x - K_i x K_i^{-1} E_i \\ &= E_i x - x E_i. \end{aligned}$$

its follows that $E_i x = x E_i$. In the same way we have $F_i x = x F_i$. Then x lies in the center. □

So we can determine the center at t generic by lifting the center of the algebra at $t = 0$.

5.2. **The center of $\mathcal{U}_\epsilon^t$.** We want to study the restriction of the deformation at the center of $\mathcal{U}_\epsilon(\mathfrak{p})$. Since t is without torsion, it is easy to see that,

Proposition 22. *i $Z_{\epsilon,0} := Z_\epsilon(\mathfrak{p})[t]/tZ_\epsilon(\mathfrak{p})[t] = Z_\epsilon(\mathfrak{p})[t]/t\mathcal{U}_\epsilon^t \cap Z_\epsilon(\mathfrak{p})[t]$.*

ii $Z_{\epsilon,0} \cong \mathbb{Z}_\epsilon(\mathfrak{p})$.

We want to prove that

Theorem 11.

$$Z_{\epsilon,0} = C_\epsilon(\mathfrak{p}).$$

Note that $Z_{\epsilon,0}$ and C_ϵ are integrally closed domain and finitely generated $Z_0(\mathfrak{p})$ algebras such that:

Lemma 15.
$$Q(Z_{\epsilon,0}) = Q(C_\epsilon)$$
where $Q(Z_{\epsilon,0})$ and $Q(C_\epsilon)$ are the fields of fractions of $Z_{\epsilon,0}$ and C_ϵ respectively.

Proof. Note that $Q(Z_{\epsilon,0}) \subset Q(C_\epsilon)$ and $Z_{\epsilon,0} \cong Z_\epsilon(\mathfrak{p})$. Since $\mathcal{U}_\epsilon(\mathfrak{p})$ and $\mathcal{S}_\epsilon(\mathfrak{p})$ have the same degree, the observation at the and of section 4.2 implies that
$$\dim_{Q(Z_0(\mathfrak{p})} Q(Z_{\epsilon,0}) = \dim_{Q(Z_0(\mathfrak{p})} Q(C_\epsilon)$$
and the result follows. □

So using classic results ([Ser65],[Mat89]) we can conclude that $Z_{\epsilon,0} = C_\epsilon$.

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