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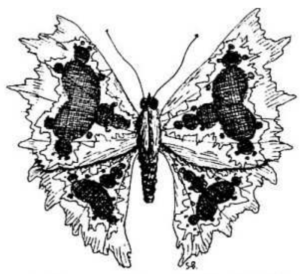
Tesi di Dottorato

Degree of Parabolic Quantum Groups



Papilio tempestae (Quantum Weather Butterfly)

Dottorando
Riccardo Pulcini

Direttore di tesi
Corrado De Concini

Coordinatrice del dottorato
Lucia Caporaso

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INTRODUCTION

Quantum groups first arose in the physics literature, particularly in the work of L.D. Faddeev and the Lenigrad school, from the "inverse scattering method", which had been developed to construct and solve "integrable" quantum systems. They have generated a great interest in the past few years because of their unexpected connections with, what are at first glance unrelated parts of mathematics, the construction of knot invariants and the representation theory of algebraic groups in characteristic p.

In their original form, quantum groups are associative algebras whose defining relations are expressed in terms of a matrix of constants (depending on the integrable system under consideration) called quantum R matrix. It was realized independently by V. G. Drinfel'd and M. Jimbo around 1985 that these algebras are Hopf algebras, which, in many cases, are deformations of "universal enveloping algebras" of Lie algebras. Indeed Drinfel'd and Jimbo give a general definition of quantum universal algebra of any semisimple complex Lie algebra. On a somewhat different case, Yu. I. Manin and S. L. Woronowicz independently studies non commutative deformations of the algebra of functions on the groups $SL_2(\mathbb{C})$ and SU_2, respectively, and showed that many of the classical results about algebraic and topological groups admit analogues in the non commutative case.

The aim of this thesis is to calculate the degree of some quantum universal enveloping algebras. Let g be a semisimple Lie algebra, fixed a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and a Borel subalgebra $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$, we denote with Δ the correspondent set of simple roots. Given $\Delta' \subset \Delta$, we associate to Δ parabolic subalgebra $\mathfrak{p} \supset \mathfrak{b}$.

Following Drifel'd, the considered situation can be quantized. We obtain Hopf algebras over $\mathbb{C}\left[q, q^{-1}\right]$, $\mathcal{U}_q(\mathfrak{b}) \subset \mathcal{U}_q(\mathfrak{p}) \subset \mathcal{U}_q(\mathfrak{g})$.

When we specialize the parameter q to a primitive l^{th} root ϵ of 1 (with some restrictions on l). The resulting algebras are finite modules over their centers, they are a finitely generated $\mathbb{C}$ algebra. In particular, every irreducible representations has finite dimension. Let us denote by $\mathcal{V}$ the set of irreducible representations, Shur lemma gives us a surjective application

$$\pi : \mathcal{V} \to \mathrm{Spec}(Z).$$

To determine the pull back of a point in $\mathrm{Spec}(Z)$ is very difficult work. But generically the problem becomes easier. Since our algebras are domain, there exists a non empty open Zariski set $V \subset \mathrm{Spec}(Z)$, such that $\pi|_{\pi^{-1}(V)}$ is bijective and moreover every irreducible representation in $\pi^{-1}(V)$ has the same dimension d, the degree of our algebra. The problem is to identify d.

Note that, a natural candidate for d exists. We will see that in the case of $\mathcal{U}_\epsilon(\mathfrak{p})$, we can find a natural subalgebra $Z_0 \subset Z$, such that as $\mathcal{U}_\epsilon(\mathfrak{p})$ subalgebra is a Hopf subalgebra. Therefore it is the coordinate ring of an algebraic group H. The deformation structure of $\mathcal{U}_\epsilon(p)$ implies that H has a Poisson structure. Let δ be the maximal dimension of the symplectic leaves, then a natural conjecture is $d = l^{\frac{\delta}{2}}$. This is well know in several cases, for example, $\mathfrak{p} = \mathfrak{g}$ and $\mathfrak{p} = \mathfrak{b}$ (cf [DCK90] and [KW76]).

Our job has been to prove this fact and to supply one explicit formula for δ.

Before describing the strategy of the proof, we explain the formula for δ. Set $\mathfrak{l}$ the Levi factor of $\mathfrak{p}$. Let $\mathcal{W}$ be the Weyl group of the root system of $\mathfrak{g}$, and $\mathcal{W}^{\mathfrak{l}} \subset \mathcal{W}$ that one of subsystem generated by Δ'. Denote by ω_0 the longest element of $\mathcal{W}$ and $\omega_0^{\mathfrak{l}}$ the longest element of $\mathcal{W}^{\mathfrak{l}}$. Recall that $\mathcal{W}$ acts on $\mathfrak{h}$ and set s as the rank of the linear transformation $w_0 - w_0^{\mathfrak{l}}$ of $\mathfrak{h}$. Then

$$\delta = l(w_0) + l(w_0^{\mathfrak{l}}) + s,$$

where l is the length function with respect to the simple reflection.

We describes now the strategy of the thesis. In order to make this, the main instrument has been the theory of quasi polynomial algebras, or skew polynomial, $\mathcal{S}$. In this case we know, following a result of De Concini, Kaç and Procesi ([DCKP92] and [DCKP95]), that to calculate the degree of $\mathcal{S}$ corresponds to calculate the rank of a matrix (with some restriction on l).

In order to take advantage of this result we have constructed a deformation of $\mathcal{U}_\epsilon(\mathfrak{p})$ to $\mathcal{S}$ and one family, $\mathcal{U}_\epsilon^{t,\chi}$, of finitely generated algebras parameterized by $(t, \chi) \in \mathbb{C} \times \mathrm{Spec}(Z_0)$. Then the fact that on $\mathbb{C}[t, t^{-1}]$ our deformations have to be trivial, together with the rigidity of semisimple algebras can be used to perform the degree computation at $t = 0$. In its the algebra we obtain is a quasi derivation algebra, so that the theorem of De Concini, Kaç and Procesi ([DCKP95]) can be applied and one is reduced to the solution of a combinatorial problem, the computation of the rank of an integer skew symmetric matrix.

The actual determination of the center of the algebra $\mathcal{U}_\epsilon(\mathfrak{p})$ remains in general an open and potentially tricky problem. However we will propose a method, inspired by work of Premet and Skryabin ([PS99]), to "left" elements of the center of the degenerate algebra at $t = 0$ to elements of the center at least over an open set of $\mathrm{Spec}\, Z_0$.

We close this introduction with the description of the chapters that compose the thesis.

In the first chapter, we recall the notion of Poisson group, this is the instrument necessary in order to describe the geometric property of the Hopf subalgebra Z_0.

In the second and third chapter, we describe the slight knowledge of algebra with trace and quasi polynomial algebra. These are the instruments necessary in order to describe the theory of the representations of $\mathcal{U}_\epsilon$ and $\mathcal{U}_\epsilon(\mathfrak{p})$. Moreover we describe how to calculate the degree of quasi polynomial algebra.

This concludes the first part of the thesis. In the second part, we will be studying the theory of quantum groups.

In chapter 4 we recall the main definitions and the main properties of quantum universal enveloping algebras associated to a semisimple complex Lie algebra $\mathfrak{g}$, and we consider how to calculate the degree of $\mathcal{U}_\epsilon(\mathfrak{g})$ and $\mathcal{U}_\epsilon(\mathfrak{b})$.

The last chapter is dedicated to the definition of quantum enveloping algebra associated to a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ and the proof of the formula for the degree.

Part I

USEFUL ALGEBRAIC AND GEOMETRIC NOTION

1. POISSON ALGEBRAIC GROUPS

In this chapter we recall some basic facts about Poisson groups that will prove useful in the study of quantum groups. The interested reader can find more details in a vast variety of articles and monographs, for example in [CP95], [KS98] or [CG97].

1.1 Poisson manifolds

1.1.1 Poisson algebras and Poisson manifolds

Definition 1.1.1. A commutative associative algebra A over a field $\mathbf{k}$ is called a *Poisson algebra* if it is equipped with a $\mathbf{k}$-bilinear operation $\{\,,\,\} : A \otimes A \to A$ such that the following conditions are satisfied:

1. A is a Lie algebra with the bracket $\{\,,\,\}$;
2. the Leibniz rule are satisfied, i.e. for any $a, b, c \in A$, we have

$$\{ab, c\} = a\,\{b, c\} + \{a, c\}\, b.$$

If these conditions are satisfied, the operation $\{\,,\,\}$ is called *Poisson bracket*, and $\xi_a = \{a, \cdot\}$ is called *Hamiltonian derivation*.

Definition 1.1.2. Let A and B be a Poisson algebra over $\mathbf{k}$. An algebra homomorphism $f : A \to B$ is called *Poisson homomorphism* if

$$f\left(\{a, b\}\right) = \{f(a), f(b)\}\,.$$

Poisson algebras form a category, with morphism being Poisson homomorphism.

Definition 1.1.3. A smooth manifold M is called a smooth *Poisson manifold* if the algebra $A = \mathcal{C}^{\infty}(M)$ of smooth complex value function on M is equipped with a structure of Poisson algebra over $\mathbb{C}$

Definition 1.1.4. An affine algebraic $\mathbf{k}$-variety M is called an *affine algebraic Poisson $\boldsymbol{k}$-variety* if the algebra $A = \mathbf{k}[M]$ of regular function on M is equipped with a structure of Poisson algebra over $\mathbf{k}$

Definition 1.1.5. 1. Let M and N be smooth Poisson manifolds. A smooth map $f : M \to N$ is called a *Poisson map* if the induced map $f^* : \mathcal{C}^\infty(N) \to \mathcal{C}^\infty(M)$ is a Poisson homomorphism.

2. Let M and N be algebraic Poisson $\mathbf{k}$-variety. An algebraic map $f : M \to N$ is called a *Poisson map* if the induced map $f^* : \mathbf{k}[N] \to \mathbf{k}[M]$ is a Poisson homomorphism.

It is clear that smooth, and algebraic, Poisson manifolds form a category, with morphisms being Poisson map.

Definition 1.1.6. Suppose that M is a smooth Poisson manifold, let $A = \mathcal{C}^\infty(M)$.

1. Given $\phi \in A$, the vector field ξ_ϕ associated to the Hamiltonian derivation $\{\phi, \}$ of A is called *Hamiltonian vector field*.

2. A submanifold (not necessarily closed) $N \subset M$ is called *Poisson submanifold* if the vector $\psi_\phi(n)$ is tangent to N for any $n \in N$ and $\phi \in A$.

Let M a smooth Poisson manifold, then there is an equivalent definition of Poisson manifold in term of bivector fields. Recall that an n-vector field is a section of the bundle $\bigwedge^n TM$ where TM is the tangent vector bundle of M. In particular, we call 2-vector fields *bivector fields*.

Recall also the definition of the *Schouten bracket* of n-vector fields which generalize the usual Lie bracket on vector fields. The Schouten bracket of an m-vector field with an n vector field is an $(m+n-1)$-vector field which is locally defined by

$$\begin{aligned}&[u_1 \wedge \ldots \wedge u_m, v_1 \wedge \ldots \wedge v_m]\\ &\qquad = \sum_{i,j} (-1)^{i+j}\, [u_i, v_j] \wedge u_1 \wedge \ldots \wedge \hat{u_i} \wedge u_m \wedge v_1 \wedge \ldots \wedge \hat{v_j} \wedge \ldots \wedge v_n\end{aligned}$$

where $u_1, \ldots, u_m, v_1, \ldots, v_n \in T_m M$, $m \in M$, and $[\,,\,]$ denote the Lie bracket of vector fields.

Denote by T^*M the cotangent bundle to M. Given a bivector field π on M, we define a bilinear operation $\{\,,\,\}$ on $\mathcal{C}^\infty(M)$ by

$$\{\phi, \psi\} = \langle \pi, d\phi \wedge d\psi \rangle \tag{1.1}$$

where $\langle\,,\,\rangle$ is the natural paring between the sections of the bundle $T^*M \wedge T^*M$ and $TM \wedge TM$.

Proposition 1.1.7. *The bracket 1.1 defines a Poisson manifold structure on M if and only if*

$$[\pi, \pi] = 0$$

Proof. Easy verification of the definition. □

Let (M, π) be a Poisson manifold. Consider the morphism of vector bundle $\breve{\pi} : T^*M \to TM$ induced by π, we have

Definition 1.1.8. A Poisson manifold is called *symplectic manifold* if the map $\breve{\pi}$ is an isomorphism.

1.1.2 Symplectic leaves

One of the most fundamental facts in the theory of Poisson manifolds is that for any Poisson manifold M there is a stratification of M by symplectic submanifolds which are called symplectic leaves in M. In a certain sense, symplectic manifolds are simple objects in the category of Poisson manifolds.

In what follows we assume that M is a smooth Poisson manifold.

Definition 1.1.9. A *Hamiltonian curve* on a smooth Poisson manifold M is a smooth curve $\gamma : [0, 1] \to M$ such that there exist $f \in \mathcal{C}^\infty(M)$ with the property that

$$\dot{\gamma}(t) = \xi_f(\gamma(t))$$

for any $t \in (0, 1)$

Definition 1.1.10. Let M be a smooth Poisson manifold.

1. We say that two points $x, y \in M$ are *equivalent* if they can be connected by a piecewise Hamiltonian curve.

2. An equivalence class of points of M is called *symplectic leaf* of M

Property. *Let S be a symplectic leaf of a smooth Poisson manifold M. Then:*

(i) S is a Poisson submanifold of M;

(ii) S is a symplectic manifold;

(iii) M is the union of its symplectic leaves.

We need a tool to determine symplectic leaves.

Definition 1.1.11. Let P_1 and P_2 be Poisson manifolds, S a symplectic manifold. A diagram $P_1 \overset{f_1}{\leftarrow} S \overset{f_2}{\rightarrow} P_2$ is called a *dual pair*, if f_1 and f_2 are Poisson maps and the Poisson subalgebras $f_1^*\mathcal{C}^\infty(P_1)$ and $f_2^*\mathcal{C}^\infty(P_2)$ of $\mathcal{C}^\infty(S)$ centralize each other with respect to the Poisson bracket.

A dual pair is called *full* if f_1 and f_2 are submersions.

Theorem 1.1.12. *Let $P_1 \overset{f_1}{\leftarrow} S \overset{f_2}{\rightarrow} P_2$ be a full dual pair. Then the blow-up $M_{x_1} = f_2 f_1^{-1}(x_1)$ is a symplectic leaf in P_2 for any $x_1 \in P_1$.*

Proof. See [KS98], page 8. □

1.2 Lie bialgebras and Manin triples

1.2.1 Lie bialgebras

Definition 1.2.1. A *Lie bialgebra* is a pair $(\mathfrak{g}, \phi)$ where $\mathfrak{g}$ is a finite dimensional Lie algebra over $\mathbf{k}$, and $\phi : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, called *cobracket*, satisfies the following conditions:

1. the dual map $\phi^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ makes $\mathfrak{g}^*$ into a Lie algebra;
2. the cobracket $\phi : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is a 1-cocycle on $\mathfrak{g}$, i.e. for any $a, b \in \mathfrak{g}$,

$$\phi([a, b]) = a.\phi(b) - b\phi(a)$$

where $a.(b \otimes c) = [a \otimes 1 + 1 \otimes a, b \otimes c] = [a, b] \otimes c + b \otimes [a, c]$.

Lie bialgebras form a category, with morphisms being Lie algebra homomorphism which commute with the cobracket.

Property. *Given a Lie bialgebra $\mathfrak{g}$, the vector space $\mathfrak{g}^*$ carries a canonical structure of Lie bialgebra, the Lie bracket being the map dual to the cobracket in $\mathfrak{g}$ and the cobracket being the map dual to the bracket in $\mathfrak{g}$.*

Definition 1.2.2. The Lie bialgebra $\mathfrak{g}^*$ is called the *dual Lie bialgebra* of $\mathfrak{g}$.

Example 1.2.3. Any Lie algebra $\mathfrak{g}$ with trivial cobracket (i.e., zero) is a Lie bialgebra.

Example 1.2.4. Let $\mathfrak{g}$ be a Lie algebra. Consider the dual vector space $\mathfrak{g}^*$ as a commutative Lie algebra. Then the map $\phi : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$ dual to the Lie bracket on $\mathfrak{g}$, define a Lie bialgebra structure on $\mathfrak{g}^*$, it is the dual Lie bialgebra to the one in example 1.2.3.

Example 1.2.5. Let $\mathfrak{g}$ be a complex simple Lie algebra with a fixed non degenerate invariant symmetric bilinear form, which is necessarily a scalar multiple of the Killig form. Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, with $n = \dim \mathfrak{h}$, the rank of $\mathfrak{g}$. Choose a set of simple roots $\alpha_1, \ldots, \alpha_n \in \mathfrak{h}^*$. This gives a decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_- \tag{1.2}$$

where $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is the nilpotent subalgebra spanned by the positive (resp. negative) root subspaces.

Let $X_i^{\pm}$, H_i, $i = 1, \ldots, n$, be the Chevalley generators corresponding to the simple root α_i, and $A = (a_{i,j})$ the Cartan matrix whose entries are $a_{i,j} = 2\frac{(\alpha_i,\alpha_j)}{(\alpha_i,\alpha_i)}$, where $(\,,\,)$ is the symmetric bilinear form on $\mathfrak{h}^*$ induced by the bilinear form on $\mathfrak{g}$.

Recall that $\mathfrak{g}$ is generated by $X_i^{\pm}$, H_i and the Serre relations

$$\begin{aligned} &[H_i, H_j] = 0, && \left[H_i, X_j^{\pm}\right] = \pm a_{i,j} X_j^{\pm}, \\ &\left[X_i^+, X_j^-\right] = \delta_{i,j} H_i, && ad^{1-a_{i,j}}\left(X_i^{\pm}\right) X_j^{\pm} = 0, \end{aligned}$$

where $ad(a)(b) = [a, b]$ is the adjoint action of $\mathfrak{g}$ on itself.

Then the following cobracket ϕ defines a Lie bialgebra structure on $\mathfrak{g}$:

$$\begin{aligned} &\phi(H_i) = 0, \\ &\phi(X_i^{\pm}) = d_i X_i^{\pm} \wedge H_i, \end{aligned}$$

where d_i, $i = 1, \ldots, n$, are positive rational numbers that satisfy $d_i a_{i,j} = d_j a_{j,i}$. For more details see [KS98].

Definition 1.2.6. The Lie bialgebra structure described in example 1.2.5 is called the *standard* Lie bialgebra structure on $\mathfrak{g}$

Example 1.2.7. Consider the simple complex Lie algebra $\mathfrak{g} = sl_2(\mathbb{C})$, then the Chevalley generator are

$$X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

then the cobracket is

$$\begin{aligned} &\phi(H) = 0, \\ &\phi(X^{\pm}) = X^{\pm} \wedge H. \end{aligned}$$

We verify only that ϕ is a cocycle.

$$\begin{aligned} \phi([H, X^{\pm}]) &= \phi(\pm 2X^{\pm}) = \pm 2X^{\pm} \wedge H \\ H \cdot \phi(X^{\pm}) - X^{\pm} \cdot \phi(H) &= H \cdot (X^{\pm} \wedge H) \\ &= [H, X^{\pm}] \wedge H = \pm 2X^{\pm} \wedge H \end{aligned}$$

$$\begin{aligned} \phi([X^+, X^-]) &= \phi(H) = 0 \\ X^+ \cdot \phi(X^-) - X^- \phi(X^+) &= X^+ \cdot X^- \wedge H - X^- \cdot X^+ \wedge H \\ &= [X^+, X^-] \wedge H + X^- \wedge [X^+, H] \\ &\quad -[X^-, X^+] \wedge H - X^+ \wedge [X^-, H] \\ &= H \wedge H + 2X^- \wedge X^+ \\ &\quad +H \wedge H + 2X^+ \wedge 2X^- \\ &= 0 \end{aligned}$$

The fact that ϕ^* defines a Lie bracket is an easy exercise.

1.2.2 Manin triples

For our proposition it is convenient to think of Lie algebras in terms of Manin triple.

Definition 1.2.8. Let $\mathfrak{g}$ be a Lie algebra equipped with a nondegenerated invariant symmetric bilinear form $\langle\, ,\, \rangle$, and g_+ and g_- Lie subalgebra of $\mathfrak{g}$. The triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is called Manin triple if:

1. $\mathfrak{g} = \mathfrak{g}_+ \oplus g_-$,
2. $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are maximal isotropic subspaces with respect to $\langle\, ,\, \rangle$.

Suppose that $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple. Since both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are maximal isotropic subspaces of $\mathfrak{g}$, we can identify $\mathfrak{g}_\mp$ with $\mathfrak{g}_\pm^*$ as a vector space.

Theorem 1.2.9. *(i) Suppose that $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ is a Manin triple. The n $(\mathfrak{g}_\pm, \phi)$ is a Lie bialgebra where the cobracket $\phi : \mathfrak{g}_\pm \to \mathfrak{g}_\pm \wedge \mathfrak{g}_\pm$ is the dual map to the Lie bracket in $\mathfrak{g}_\mp$.*

(ii) Suppose that $(\mathfrak{g}, \phi)$ is a Lie bialgebra. Consider the vector space $\mathfrak{g}^*$ equipped with the dual Lie bialgebra structure (cf. definition 1.2.2). Then:

$$(\mathfrak{g} \oplus \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$$

is a Manin triple, with the Lie algebra structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ given by

$$[a + \alpha, b + \beta] = [a, b] + [\alpha, \beta] + ad^*_\alpha(b) - ad^*_\beta(a) - ad^*_b(\alpha) + ad^*_a(\beta),$$

for any a, $b \in \mathfrak{g}$, α, $\beta \in \mathfrak{g}^*$, and the bilinear form on $\mathfrak{g} \oplus \mathfrak{g}^*$ given by

$$\langle a + \alpha, b + \beta\rangle = \beta(a) + \alpha(b).$$

Note that, in particular

$$[\alpha, b] = ad^*_\alpha(b) - ad^*_b(\alpha),$$
$$[a, \beta] = ad^*_a(\beta) - ad^*_\beta(a).$$

Proposition 1.2.10. *Let $\mathfrak{g}$ be a Lie bialgebra, and $\mathcal{D}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ the lie algebra describe in theorem 1.2.9. Then there exist a canonical Lie bialgebra structure on $\mathcal{D}(\mathfrak{g})$ given by the cobracket*

$$\phi_{\mathcal{D}(\mathfrak{g})}\,(a + \alpha) = \phi_\mathfrak{g}(a) + \phi_{\mathfrak{g}^*}(\alpha),$$

for any $a \in \mathfrak{g}$ and $\alpha \in \mathfrak{g}^*$

Definition 1.2.11. Lie bialgebra $\mathcal{D}(\mathfrak{g})$ is called *the double Lie bialgebra* (sometimes it is called *the classical double* of $\mathfrak{g}$)

Remark 1.2.12. Manin triple which corresponds to the double Lie bialgebra is

$$(\mathcal{D}(\mathfrak{g}) \oplus \mathcal{D}(\mathfrak{g}), \mathcal{D}(\mathfrak{g}), \mathfrak{g} \oplus \mathfrak{g}^*)$$

where $\mathcal{D}(\mathfrak{g})$ is embedded into $\mathcal{D}(\mathfrak{g}) \oplus \mathcal{D}(\mathfrak{g})$ as the diagonal, and $\mathfrak{g} \oplus \mathfrak{g}^*$ (the dual Lie bialgebra of $\mathcal{D}(\mathfrak{g})$) is the direct sun of the Lie algebras $\mathfrak{g} = (\mathfrak{g}, 0) \subset \mathcal{D}(\mathfrak{g}) \oplus \mathcal{D}(\mathfrak{g})$ and $\mathfrak{g}^* = (0, \mathfrak{g}^*) \subset \mathcal{D}(\mathfrak{g}) \oplus \mathcal{D}(\mathfrak{g})$. The bilinear form is given by

$$\langle (a, b), (c, d) \rangle = \langle a, c \rangle_{\mathcal{D}(\mathfrak{g})} - \langle b, d \rangle_{\mathcal{D}(\mathfrak{g})},$$

for any $a, b, c, d \in \mathcal{D}(\mathfrak{g})$.

Double Lie bialgebra has an important property. Namely, the cobracket on such Lie bialgebra is described by a very simple formula, as shown in the following proposition

Proposition 1.2.13. *Let $\mathfrak{g}$ be a Lie bialgebra, and $\mathcal{D}(\mathfrak{g})$ the double Lie bialgebra. Suppose that $\{e_\alpha\}$ is a basis of $\mathfrak{g}$ and $\{e^\alpha\}$ is the dual basis of $\mathfrak{g}^*$. Let us identify e_α with $(e_\alpha, 0) \in \mathcal{D}(\mathfrak{g})$ and e^α with $(0, e^\alpha) \in \mathcal{D}(\mathfrak{g})$. Then the cobracket in $\mathcal{D}(\mathfrak{g})$ is given by*

$$\phi(a) = [a \otimes 1 + 1 \otimes a, r]$$

for any $a \in \mathcal{D}(\mathfrak{g})$, where

$$r = \sum_\alpha e_\alpha \otimes e^\alpha \in \mathcal{D}(\mathfrak{g}) \otimes \mathcal{D}(\mathfrak{g}),$$

is the canonical element related to $\langle\, , \rangle$.

We give now some important examples of Lie bialgebra and corresponding Manin triple.

Example 1.2.14. Let $\mathfrak{g}$ be a complex simple Lie algebra equipped with the standard Lie bialgebra structure (cf. definition 1.2.5) related to a fixed invariant bilinear form $\langle\, , \rangle$, let $\mathfrak{h}$ be the corresponding Cartan subalgebra and define $\mathfrak{b}_\pm = \mathfrak{n}_\pm \oplus \mathfrak{h}$ a Borel subalgebra of $\mathfrak{g}$. Then the corresponding Manin triple is $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}, \mathfrak{s})$, where $\mathfrak{g}$ is embedded diagonally into $\mathfrak{g} \oplus \mathfrak{g}$, and

$$\mathfrak{s} = \{(x, y) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- : x_\mathfrak{h} + y_\mathfrak{h} = 0\},$$

where $x_\mathfrak{h}$ denote the "Cartan part" of $x \in \mathfrak{b}_\pm$. In other words, $x = x_\pm + x_\mathfrak{h}$ where $x_\pm \in \mathfrak{n}_\pm$ and $x_\mathfrak{h} \in \mathfrak{h}$. The invariant bilinear form $\mathfrak{g} \oplus \mathfrak{g}$ is given by

$$\langle (a, b), (c, d) \rangle = \langle a, c \rangle_\mathfrak{g} - \langle b, d \rangle_\mathfrak{g}.$$

Note that as Lie algebra $\mathcal{D}(\mathfrak{g})$ is isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$. For more details one can see for example [KS98].

Example 1.2.15. Let $\mathfrak{g}$ be as in the previous example. Consider the Borel subalgebras $\mathfrak{b}_{\pm}$, note that they are in fact Lie bialgebras with respect to the restriction of the cobracket from $\mathfrak{g}$ to $\mathfrak{b}_{\pm}$. Then the corresponding Manin triple is

$$(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{b}_+, \mathfrak{b}_-),$$

where the Borel subalgebras are embedded into $\mathfrak{g} \oplus \mathfrak{h}$ so that if $a \in \mathfrak{b}_+$ then $a \to (a, a_{\mathfrak{h}})$ and if $a \in \mathfrak{b}_-$ then $a \to (a, -a_{\mathfrak{h}})$, where we use the notation in the previous example, the bilinear form is given by

$$\langle (a,b),(c,d)\rangle = \langle a,c\rangle_{\mathfrak{g}} - \langle b,d\rangle_{\mathfrak{h}}.$$

with a, $c \in \mathfrak{g}$ and b, $d \in \mathfrak{h}$, and $\langle\,,\,\rangle_{\mathfrak{h}}$ as the restriction of $\langle\,,\,\rangle_{\mathfrak{g}}$ to $\mathfrak{h}$. In particular, we see that the Borel subalgebras $\mathfrak{b}_+$ and $\mathfrak{b}_-$ are dual to each other as Lie bialgebra.

We consider, in the next example, an intermediate case between Borel subalgebra $\mathfrak{b}$ and Lie algebra $\mathfrak{g}$.

Example 1.2.16. Using the notation of the previous example, we call parabolic subalgebra any subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ such that $\mathfrak{b} \subseteq \mathfrak{p}$, note that $\mathfrak{b}_{\pm}$ and $\mathfrak{g}$ are examples of parabolic subalgebra. Set R as the set of rot associated to $\mathfrak{g}$ and X_α $\alpha \in R$ the root vectors, we define $R_{\mathfrak{p}} := \{\alpha \in R : X_\alpha \in \mathfrak{p}\}$. We call $\mathfrak{l} := \mathfrak{h} \oplus \bigoplus_{\alpha \in R_{\mathfrak{p}} : -\alpha \in R_{\mathfrak{p}}} \mathbb{C}X_\alpha$ the Levi factor and $\mathfrak{u} = \bigoplus_{\alpha \in R_{\mathfrak{p}} : -\alpha \notin R_{\mathfrak{p}}} \mathbb{C}X_\alpha$ the unipotent part of $\mathfrak{p}$. Moreover, we have $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$, for more details one can see [Bou98] or [Hum78].

Proposition 1.2.17. $\mathfrak{p}$ *is a Lie bialgebra with respect to the restriction of the cobracket from* $\mathfrak{g}$ *to* $\mathfrak{p}$.

Proof. Simple verification of the definition. □

Proposition 1.2.18. *The Manin triple corresponding to* $\mathfrak{p}$ *is*

$$(\mathfrak{g} \oplus \mathfrak{l}, \mathfrak{p}, \mathfrak{s}),$$

where $\mathfrak{p}$ *is embedded into* $\mathfrak{g} \oplus \mathfrak{l}$ *so that* $a \to (a, a_{\mathfrak{l}})$ *and*

$$\mathfrak{s} = \left\{ (x,y) \in \mathfrak{b}_- \oplus \mathfrak{b}_+^{\mathfrak{l}} : x_{\mathfrak{h}} + y_{\mathfrak{h}} = 0 \right\},$$

where $\mathfrak{b}_+^{\mathfrak{l}} = \mathfrak{b}_+ \cap \mathfrak{l}$, *the bilinear form is given by*

$$\langle (a,b),(c,d)\rangle = \langle a,c\rangle_{\mathfrak{g}} - \langle b,d\rangle_{\mathfrak{l}}.$$

with a, $c \in \mathfrak{g}$ *and* b, $d \in \mathfrak{l}$, *and* $\langle\,,\,\rangle_{\mathfrak{l}}$ *is the restriction of* $\langle\,,\,\rangle_{\mathfrak{g}}$ *to* $\mathfrak{l}$.

Proof. Observe that, the following decomposition hold in $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{u}_- \oplus \mathfrak{l} \oplus \mathfrak{u}$$

with $\mathfrak{u}_- := \bigoplus_{\alpha \in R \setminus R_{\mathfrak{p}}}$, and

$$\dim(\mathfrak{g} \oplus \mathfrak{l}) = \dim(\mathfrak{g}) + \dim(\mathfrak{l}) = \dim(\mathfrak{p}) + u + \dim(\mathfrak{l}) = 2\dim(\mathfrak{p})$$

where $u = \dim \mathfrak{u}_- = \dim \mathfrak{u}_+$ Let us now check that $\mathfrak{p} \cap \mathfrak{s} = \{0\}$, take $(a, b) \in \mathfrak{p} \cap \mathfrak{s}$, then

- $(a, b) \in \mathfrak{p} \Rightarrow b = a_{\mathfrak{l}}$
- $(a, b) \in \mathfrak{s} \Rightarrow a \in \mathfrak{b}_-$, $b \in \mathfrak{b}_+^{\mathfrak{l}}$ and $a_{\mathfrak{h}} + b_{\mathfrak{h}} = 0$

Then $b = t \in \mathfrak{h} = \mathfrak{b}_+ \cap \mathfrak{b}_-$, and $a = u + t$ with $u \in \mathfrak{u} \subset \mathfrak{b}_+$, but $a \in \mathfrak{b}_-$, so it follows that $u = 0$ and $a = b = t$. Then the last condition gives us

$$0 = a + b = 2t \Rightarrow t = 0.$$

In conclusion we have $a = b = 0$. Hence $\mathfrak{p} \cap \mathfrak{s} = \{0\}$. Observe that

$$\dim \mathfrak{s} = \dim \mathfrak{b}_+ + \dim \mathfrak{b}_-^{\mathfrak{l}} - \dim \mathfrak{h} = \dim \mathfrak{p},$$

then we have

$$\mathfrak{g} \oplus \mathfrak{l} = \mathfrak{p} \oplus \mathfrak{s}.$$

It remains to show that $\mathfrak{p}$ and $\mathfrak{s}$ are isotropic subspace with respect to $\langle\, ,\, \rangle$.

Recall that $\langle\, ,\, \rangle_{\mathfrak{g}}$ is an invariant symmetric bilinear form for $\mathfrak{g}$, note that for any parabolic subalgebra $\mathfrak{b}_+ \subseteq \mathfrak{q} \subseteq \mathfrak{g}$, its Levi factor $\mathfrak{m}$ we have:

$$\langle x, y \rangle_{\mathfrak{g}} = \langle x_{\mathfrak{h}}, y_h \rangle_{\mathfrak{m}}$$

for every x, $y \in \mathfrak{q}$.

Let $(a, b) \in \mathfrak{p}$ then $b = a_{\mathfrak{l}}$, and we have

$$\begin{aligned} \langle (a,b), (a,b) \rangle &= \langle a, a \rangle_{\mathfrak{g}} - \langle b, b \rangle_{\mathfrak{l}} \\ &= \langle a, a \rangle_{\mathfrak{g}} - \langle a_{\mathfrak{l}}, a_{\mathfrak{l}} \rangle_{\mathfrak{l}} \\ &= 0. \end{aligned}$$

Let $(a, b) \in \mathfrak{s}$ then $a_h + b_h = 0$. Recall that the Levi factor for $b_\pm$ is $\mathfrak{h}$. We get

$$\begin{aligned} \langle (a,b), (a,b) \rangle &= \langle a, a \rangle_{\mathfrak{g}} - \langle b, b \rangle_{\mathfrak{l}} \\ &= \langle a_{\mathfrak{h}}, a_{\mathfrak{h}} \rangle_{\mathfrak{h}} - \langle b_{\mathfrak{h}}, b_{\mathfrak{h}} \rangle_{\mathfrak{h}} \\ &= \langle a_{\mathfrak{h}}, a_{\mathfrak{h}} \rangle_{\mathfrak{h}} - \langle -a_{\mathfrak{h}}, -a\mathfrak{h} \rangle_{\mathfrak{h}} \\ &= \langle a_{\mathfrak{h}}, a_{\mathfrak{h}} \rangle_{\mathfrak{h}} - \langle a_{\mathfrak{h}}, a_{\mathfrak{h}} \rangle_{\mathfrak{h}} \\ &= 0. \end{aligned}$$

This finished the proof that $(\mathfrak{g} \oplus \mathfrak{l}, \mathfrak{p}, \mathfrak{s})$ is the Manin triple associated to $\mathfrak{p}$. $\square$

1.3 Poisson groups

1.3.1 Poisson affine algebraic group

Let G an affine algebraic groups over an algebraic closed field $\mathbf{k}$. Let $\mathbf{k}[G]$ its coordinate ring. We know that $\mathbf{k}[G]$ is a Hopf algebra with

$$\begin{aligned} \text{Comultiplication:} \quad & \Delta : \quad \mathbf{k}[G] \to \mathbf{k}[G] \otimes \mathbf{k}[G] \\ \text{Antipode:} \quad & S : \quad \mathbf{k}[G] \to \mathbf{k}[G] \\ \text{Counit:} \quad & \epsilon : \quad \mathbf{k}[G] \to \mathbf{k} \end{aligned}$$

given respectively by $\Delta(f)(h_1, h_2) = f(h_1 h_2)$, $\forall\ h_1,\ h_2 \in G$, $S(f)(h) = f(h^{-1})$, $\forall\ h \in G$, $\epsilon(f) = f(e)$ where $e \in G$ is the identity element.

Definition 1.3.1. Suppose that a Hopf algebra A is equipped with a Poisson algebra structure. We say that A is a *Poisson Hopf algebra* if both structure are compatible in the sense that the comultiplication Δ is a Poisson algebra homomorphism, where the Poisson structure on $A \otimes A$ is given by

$$\{a \otimes b, c \otimes d\} = \{a, c\} \otimes bd + ac \otimes \{b, d\}$$

Property. *Let A be a Poisson Hopf algebra, then the counit ϵ is a Poisson algebra homomorphism, and the antipode S is a Poisson algebra antiautomorphism.*

Proof. See [KS98], page 18. □

We know that the algebra $\mathbf{k}[G]$ of regular functions on an algebraic group G is a Hopf algebra. Recall that it is also a Poisson algebra if G is a Poisson algebraic variety.

Definition 1.3.2.
1. Suppose G is an algebraic group over the field $\mathbf{k}$ equipped with a Poisson manifold structure. We say that G is a *Poisson algebraic group* if the algebra $\mathbf{k}[G]$ of regular functions on G is a Poisson Hopf algebra.
2. Equivalently, G is a *Poisson algebraic group* if the multiplication $m : G \times G \to G$ is a Poisson map.

Property. *Let G a Poisson algebraic group, then the map $s : G \to G$, such that $s(g) = g^{-1}$ for every $g \in G$, is an anti-Poisson map, i.e. the dual map $s^* : \boldsymbol{k}[G] \to \boldsymbol{k}[G]$ is an antihomomorphism of Poisson map.*

Note. Poisson algebraic groups form a category, with morphisms being homomorphisms which are the same time Poisson maps.

1.3.2 Poisson Lie group

The definition of Poisson algebraic group formulated in the language of points can be easily carried over to the case of Poisson Lie groups. The principal difficulty is that the comultiplication maps goes into $\mathcal{C}^\infty(G \times G)$ but not necessarily into $\mathcal{C}^\infty(G) \otimes \mathcal{C}^\infty(G) \neq \mathcal{C}^\infty(G \times G)$

Definition 1.3.3. Let G be a Lie group and at the same time, a Poisson manifold. We say that G is a *Poisson Lie Group* if the multiplication $m : G \times G \to G$ is a Poisson map.

Note. Poisson Lie groups form a category, with morphisms being homomorphisms which are the same time Poisson maps.

Proposition 1.3.4. *Let G be both a Lie group and a Poisson manifold, and let π be the bivector field corresponding to the Poisson manifold structure on G. Then G is a Poisson Lie group if and only if π satisfies the following condition:*

$$\pi(g_1 g_2) = (l_{g_1})_* \pi(g_2) + (r_{g_2})_* \pi(g_1)$$

for any g_1, $g_2 \in G$, where l_g is the left multiplication and r_g is the right multiplication.

Proof. See [KS98], page 19. □

Corollary 1.3.5. *The unit element of a Poisson Lie group is always a zero-dimensional symplectic leaf*

In this thesis we always used Poisson algebraic groups so we shall describe Poisson brackets on the algebras of regular functions.

Example 1.3.6. Every Lie group G with trivial Poisson bracket is a Poisson Lie group.

Example 1.3.7. Consider the abelian Lie group $G = \mathbb{C}^n$. Note that by linearity it suffices to define the Poisson bracket on the coordinate function. To get a Poisson Lie group structure we can take:

$$\{x_i, x_j\} = \sum_{k=1}^{n} c_{i,j}^k x_k$$

where x_m, $m = 0 \ldots n$, are the coordinate function, and the structure constant $c_{i,j}^k$ satisfy the following condition:

$$\begin{aligned} c_{i,j}^k &= -c_{j,i}^k \\ \sum_{l=1}^{n} \left(c_{i,j}^l c_{l,k}^m + c_{j,k}^l c_{l,i}^m + c_{k,i}^l c_{l,j}^m \right) &= 0 \end{aligned}$$

Example 1.3.8. An important special case of the previous example is The Kirillov-Kostant bracket. Let $\mathfrak{g}$ a lie algebra. Consider the dual vector space $\mathfrak{g}^*$ equipped with the structure of abelian Lie group. The Kirillov-Kostant bracket on $\mathfrak{g}^*$ is given by:

$$\{a, b\} = [a, b]_{\mathfrak{g}}$$

where a, $b \in \mathfrak{g}$ are regarded as linear function on $\mathfrak{g}^*$.

Example 1.3.9. Consider the Lie group $G = SL_2(\mathbb{C})$ of complex 2×2 matrices with determinant 1. Then the following relations define a Poisson Lie group structure on G:

$$\begin{aligned}
\{t_{11}, t_{12}\} &= -t_{11}t_{12}, \\
\{t_{11}, t_{21}\} &= -t_{11}t_{21}, \\
\{t_{12}, t_{22}\} &= -t_{12}t_{22}, \\
\{t_{21}, t_{22}\} &= -t_{21}t_{22}, \\
\{t_{12}, t_{21}\} &= 0, \\
\{t_{11}, t_{22}\} &= -2t_{12}t_{21},
\end{aligned}$$

where t_{ij}, $(i, j = 1, 2)$, are the matrix elements.

1.3.3 The correspondence between Poisson Lie groups and Lie bialgebras

One of the most important facts of the Lie theory is the correspondence between Lie groups and Lie algebras. Recall that given a Lie group G, the tangent space at the unit element has a canonical Lie algebra structure. Conversely, given a Lie algebra $\mathfrak{g}$, there exists a unique connected and simple connected Lie group whose tangent space at the unit element is isomorphic to $\mathfrak{g}$ as Lie algebra.

We establish now a Poisson counterpart of this result. Let G a Poisson Lie group, and $\mathfrak{g}$ its Lie algebra. As usual identify $\mathfrak{g}$ with the tangent space T_eG to G at the unit element of the group. Define a linear map $\phi : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ as the linear map which is dual to the Lie bracket in $\mathfrak{g}^* = T_e^*G$ given by

$$[\alpha, \beta] = d_e \{f, g\} \tag{1.3}$$

for any α, $\beta \in \mathfrak{g}^*$, where f, $g \in \mathcal{C}^\infty(G)$ are such that

$$d_e f = \alpha,\ d_e g = \beta.$$

Theorem 1.3.10. *(i) Let G be a Poisson Lie group. Then there exists a canonical Lie bialgebra structure on the Lie algebra $\mathfrak{g} = T_eG$ with the cobracket $\phi : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ given by 1.3.*

(ii) Let G be a Lie group, and suppose that the Lie algebra $\mathfrak{g} = T_eG$ is equipped with a Lie bialgebra structure. Then there exist a unique Poisson Lie group structure on G such that 1.3 holds.

Proof. See [KS98], page 21. □

Proposition 1.3.11. *(i) The correspondence $G \mapsto \mathfrak{g} = T_e G$ established a covariant functor from the category of the Poisson Lie groups to the category of Lie bialgebras.*

(ii) The correspondence establishes an equivalence between the full subcategory of connected and simply connected Poisson Lie groups and the category of Lie bialgebras.

Example 1.3.12. Let G be a Lie group with trivial Poisson structure then $\mathfrak{g}$ is a Lie bialgebra with trivial cobracket

Example 1.3.13. Let G be the Lie group with Kostant-Kirillov structure, define in example 1.3.8, then $\mathfrak{g}$ is the Lie bialgebra given in the example 1.2.4

Definition 1.3.14. Let G be a Poisson Lie group, $\mathfrak{g}$ the Lie bialgebra of G.

1. A connected Poisson Lie group G^* which corresponds to the Lie bialgebra $\mathfrak{g}^*$ is called a *dual Poisson Lie group*.

2. A connected Poisson Lie group $\mathcal{D}(G)$ which corresponds to the double Lie bialgebra $\mathcal{D}(\mathfrak{g})$ is called a *double Poisson Lie group* of G

1.4 Symplectic leaves in Poisson groups

1.4.1 Symplectic leaves in Poisson Lie groups and dressing action

Now we use theorem 1.1.12 to describe symplectic leaves in a Poisson Lie group. The double Poisson Lie group construction allows us to describe the symplectic leaves locally as the orbits of the so-called dressing action. Throughout the section the word "locally" means "in a neighborhood of the unit element of the group".

Let G be a Poisson Lie group, and $\mathfrak{g}$ the Lie bialgebra of G. Recall that we have a simple description of the dual Lie bialgebra structure on $\mathcal{D}(\mathfrak{g})$ by

$$\phi(a) = [a \otimes 1 + 1 \otimes a, r]$$

for any $a \in \mathcal{D}(\mathfrak{g})$, where $r = \sum_\alpha e_\alpha \otimes e^\alpha$, $\{e_\alpha\}$ being a basis of $\mathfrak{g}$ and $\{e^\alpha\}$ the dual basis of $\mathfrak{g}^*$.

Proposition 1.4.1. *The Poisson bracket on $\mathcal{D}(G)$ is given by*

$$\{f_1, f_2\} = \sum_\alpha (\delta_\alpha f_1 \delta^\alpha f_2 - \delta^\alpha f_1 \delta_\alpha f_2) - \sum_\alpha \left(\delta'_\alpha f_1 (\delta^\alpha)' f_2 - (\delta^\alpha)' f_1 \delta'_\alpha f_2\right)$$

where δ_α (resp. δ'_α) is the right (resp. the left) invariant vector field on $\mathcal{D}(G)$ which takes the value e_α at the unit element of $\mathcal{D}(G)$, while δ^α (resp. (δ^α)') is the right (resp. the left) invariant vector field on $\mathcal{D}(G)$ which takes the value e^α at the unit element of $\mathcal{D}(G)$.

Proof. See [KS98], page 23. □

Proposition 1.4.2. *The multiplication maps*

$$m \quad : \quad G \times G^* \to \mathcal{D}(G), \tag{1.4}$$
$$m \quad : \quad G^* \times G \to \mathcal{D}(G) \tag{1.5}$$

are local Poisson diffeomorphisms in a neighborhood of the unit element.

Proof. This is an easy consequence of the fact that $\mathcal{D}(\mathfrak{g}) = \mathfrak{g} \oplus \mathfrak{g}^*$ as a vector space. □

It is clear that one can identify locally G with $G^* \backslash \mathcal{D}(G)$ or with $\mathcal{D}(G)/G^*$. Consider the natural projection $p_1 : \mathcal{D}(G) \to G^* \setminus \mathcal{D}(G)$ and $p_2 : \mathcal{D}(G) \to \mathcal{D}(G)/G^*$. Then we have the following property

Proposition 1.4.3. *The Poisson manifold structure on the double Lie group $\mathcal{D}(G)$ induces Poisson manifold structure on $G^* \setminus \mathcal{D}(G)$ and $\mathcal{D}(G)/G^*$ such that the natural projections p_1 and p_2 are Poisson maps. Both manifolds are isomorphic to G as Poisson manifolds in a neighborhood of the coset G^*.*

Now we introduce the notion of left and right dressing actions. First we define them locally. Given $g \in G$ and $h \in G^*$ which lie in some neighborhoods of the unit element of $\mathcal{D}(G)$, using proposition 1.4.2 there exist unique $g^h \in G$ and $h^g \in G^*$ such that

$$hg = g^h h^g \tag{1.6}$$

This formula defines a local left action of G^* on G and a local right action of G on G^* given by

$$h : g \mapsto g^h, \; g : h \mapsto h^g \tag{1.7}$$

respectively. Note that we can replace G by G^* and vice versa, so that we have also a local right action of G^* on G and a local left action of G on G^*.

Definition 1.4.4. The local left (risp. right) action of G on G^* define by 1.6 and 1.7 is called *local left* (risp. *right*) *dressing action* of G on G^*. If it can be extended to a global action, the latter is called *global left* (resp. *right*) *dressing action* of G on G^*.

Proposition 1.4.5. *Let G a Poisson Lie group and G^* the Poisson Lie group dual to G. Then the symplectic leaves in G locally coincide with the orbits of the right (or left) dressing action of G^*. In particular, if the right (resp. left) dressing action is defined globally, the symplectic leaves are the orbits of the right (risp. left) dressing action.*

Proof. See [KS98], page 26. □

Theorem 1.4.6. *Let $g \in G$ belong to a sufficiently small neighborhood of the unit element in G. The symplectic leaf in G passing trough g locally (in some neighborhood of g) is the image of the double coset $G^* g G^* \subset \mathcal{D}(G)$ under the natural projection $\mathcal{D}(G) \to \mathcal{D}(G)/G^* \backsimeq G$*

Proof. See [KS98], page 27. □

1.4.2 Symplectic leaves in simple complex Poisson Lie groups

Let G be a finite dimensional simple complex Lie group, and $\mathfrak{g}$ the Lie algebra of G. Suppose that $\mathfrak{g}$ is equipped with the standard bialgebra structure described in the example 1.2.5. It induces a Poisson Lie group structure on G which is also called standard. Our goal is to describe the symplectic leaves in G.

Let $(\mathfrak{g} \oplus \mathfrak{g}, \mathfrak{g}, \mathfrak{s})$ the Manin triple associated to $\mathfrak{g}$, described in the example 1.2.14. It is easy to see that $\mathcal{D}(G)$ is isomorphic to $G \times G$ as Lie group and we can choose $G^* = \{(g_1, g_2) : g_1 \in B_+,\ g_2 \in B_- \text{ and } (g_1)_H (g_2)_H = e\}$ as dual Poisson group. It is also clear that $G \cdot G^*$ is dense and open in $\mathcal{D}(G)$. Moreover, the multiplication map $m : G \times G^* \to G \cdot G^*$ is a covering space.

We conclude that the image $\tilde{G}$ of the quotient map $p : G \to \mathcal{D}(G)/G^*$ is dense and open in $\mathcal{D}(G)/G^*$. By theorem 1.1.12 and theorem 1.4.6 we get

Lemma 1.4.7. *(i) Any symplectic leaf in G is a connected component of the fiber $p^{-1}(S)$, where S is a symplectic leaf in $\tilde{G}$*

(ii) Any symplectic leaf in $\tilde{G}$ is of the form $\tilde{G} \cap G^* g G^*/G^*$ for some $g \in G \subset \mathcal{D}(G)$.

Now we want to describe the symplectic leaves. It appears that they are related to the Bruhat decomposition.

Let $b_\pm = \mathfrak{n}_\pm \oplus \mathfrak{h}$ be the Borel subalgebra of $\mathfrak{g}$, and $\mathcal{B}_\pm$ the corresponding Borel subgroup of G. Recall that the Weyl group $\mathcal{W}$ is generated by the simple reflection $s_i : \mathfrak{h}^* \to \mathfrak{h}^*$ given by:

$$s_i(\lambda) = \lambda - \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i}\alpha_i$$

where $\alpha_1, \ldots, \alpha_r$ are the simple root, and $r = \dim \mathfrak{h}$ is the rank of $\mathfrak{g}$. The following results are well know (cf. [Ste68])

Proposition 1.4.8. *The following decomposition of G holds:*

$$G = \bigcup_{\omega \in \mathcal{W}} B_\pm \omega B_\pm \tag{1.8}$$

It is called the Bruhat decomposition *of G*

Let $X = G/B_{\pm}$ the flag manifold. By proposition 1.4.8, we have

$$X = \bigcup_{\omega \in \mathcal{W}} X_\omega \text{ where } X_\omega \backsimeq B_{\pm}\omega B_{\pm}/B_{\pm}$$

Definition 1.4.9. X_ω is the so-called *Schubert cell* of X corresponding to $\omega \in \mathcal{W}$.

It is well know that X_ω is naturally isomorphic to $\mathbb{C}^{l(\omega)}$, where $l(\omega)$ is the length of ω.

Proposition 1.4.10. *The following Bruhat decomposition holds:*

$$\mathcal{D}(G) = \bigcup_{(\omega_1,\omega_2)\in\mathcal{W}\times\mathcal{W}} P \cdot (\omega_1, \omega_2) \cdot P, \tag{1.9}$$

where $P = HG^*$ and H is the distinguished Cartan subgroup of G associated to $\mathfrak{h}$.

Consider the following sets:

$$\begin{aligned} C_{(\omega_1,\omega_2)} &= (G^* \cdot (\omega_1, \omega_2) \cdot G^*)/G^*, \\ B_{(\omega_1,\omega_2)} &= C_{(\omega_1,\omega_2)} \cap \tilde{G}, \\ A_{(\omega_1,\omega_2)} &= p^{-1}\left(\bigcup_{h\in H} hB_{(\omega_1,\omega_2)}\right) \end{aligned}$$

Proposition 1.4.11. *(i) each symplectic leaf in $\tilde{G}$ is of the form $hB_{(\omega_1,\omega_2)}$ for some $h \in H$ and $(\omega_1, \omega_2) \in \mathcal{W} \times \mathcal{W}$.*

(ii) each symplectic leaf in G is of the form $hA_{(\omega_1,\omega_2)}$ for some $h \in H$ and $(\omega_1, \omega_2) \in \mathcal{W} \times \mathcal{W}$.

Proposition 1.4.12. *Denote $s(\omega_1, \omega_2) = \operatorname{codim}_{\mathfrak{h}} \ker(\omega_1\omega_2^{-1} - 1)$. Then*

$$C_{(\omega_1,\omega_2)} \backsimeq H^{s(\omega_1,\omega_2)} \times \mathbb{C}^{l(\omega_1)+l(\omega_2)}$$

Example 1.4.13. The following is the full list of the symplectic leaves in $SL_2(\mathbb{C})$:

$$\begin{aligned} T_t &= \left\{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\right\}, t \neq 0, \\ T_tA_{(e,\omega_0)} &= \left\{\begin{pmatrix} t & b \\ 0 & t^{-1} \end{pmatrix} : b \neq 0\right\}, t \neq 0, \\ T_tA_{(\omega_0,e)} &= \left\{\begin{pmatrix} t & 0 \\ c & t^{-1} \end{pmatrix} : c \neq 0\right\}, t \neq 0, \\ T_tA_{(\omega_0,\omega_0)} &= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : b, c \neq 0, \frac{b}{c} = t^2\right\}, t \neq 0. \end{aligned}$$

Finally, consider the map $\mu : G^* \to G$ given by $\mu((b_1, b_2)) = b_1^{-1}b_2$, so that μ is a covering of the big Bruhat cell B^-B^+. De Concini, Kaç and Procesi shows in [DCKP92], that as soon as $C \subset G$ is a conjugacy class, until $\dim C > 0$, $\mu^{-1}(C) \subset G^*$ is a single symplectic leaf of G^*. If $\dim C = 0$, i.e. $C = \{x\}$ is an element of the center of G, then $\mu^{-1}(x)$ has a finite number of elements each of which is a symplectic leaf.

2. ALGEBRAS WITH TRACE

In this chapter we will require some slight knowledge of the theory of algebras with trace that will be useful, in the next chapters, for the study of quantum groups and quasi polynomial algebras. More details and a more general approach in order to study these algebras can be found in [Pro87], [Pro73], [Pro74] or [Pro79].

2.1 Definition and properties

Let A be an associative algebra with an unit element 1 over a field $\mathbf{k}$ of characteristic 0 and let us denote the algebraic closure of $\mathbf{k}$ by $\overline{\mathbf{k}}$.

Definition 2.1.1. A *trace map* in an algebra A is a linear map

$$tr : A \to A$$

satisfying the following axioms: for all pairs of element a, $b \in A$

1. $tr(ab) = tr(ba)$
2. $tr(a)b = btr(a)$
3. $tr(tr(a)b) = tr(a)tr(b)$

An algebra with a trace map is called *algebra with trace*

Note. The value of the trace is a subalgebra of the center of A (by condition 2).

An ideal I of A algebra with trace is an ordinary ideal closed under trace, so that A/I inherits a trace.

We are interested in a particular family of algebra with trace as in [Pro87]. Once we have a trace map we want to define for all $a \in A$ the element $\sigma_k(a)$ "the symmetric function over the eigenvalue of a", by declaring that $tr(a^k)$ should be the sum of the k^{th} power of the eigenvalues of a. To do this recall that in the ring $\mathbb{Q}[x_1, \dots, x_n]$ it defines the elementary symmetric function by the identity

$$\prod(t - x_i) = \sum_{i=0}^{n}(-1)^i \sigma_i t^{d-i}$$

and the power sums function $\psi_k = \sum x_i^k$. It is easy to prove that, for every $k \leq n$, is a polynomial $p_k(y_1, \ldots, y_k)$ with rational coefficient, independent of n and such that:

$$\sigma_k = p_k(\psi_1, \ldots, \psi_k).$$

We then set

$$\sigma_k(a) = p_k(tr(a), \ldots, tr(a^k)).$$

Next we can formally define for every element $a \in A$ and for every integer d a d^{th}-*characteristic polynomial*:

$$\chi_{d,a}[t] = \sum_{i=0}^{d} (-1)^i \sigma_i(a) t^{d-i}$$

Definition 2.1.2. 1. We say that an algebra A with trace satisfies the d^{th}-*formal Cayley-Hamilton* theorem if $\chi_{d,a}[a] = 0$ for all $a \in A$.

2. We say that A has *degree d* if it satisfies the d^{th}-formal Cayley-Hamilton theorem and $tr(1) = d$.

Note that algebra with trace of degree d form a category with morphisms being algebra morphisms compatible with trace, which will be denoted by $\mathcal{C}_d$.

2.2 Representations

We are interested in a representation theory of algebra with trace of degree d. Let give an example

Example 2.2.1. Let A be a commutative algebra, then $M_d(A)$ with the ordinary trace is an algebra with trace of degree d.

Definition 2.2.2. A n dimensional *representation* of an algebra with trace R with value in a commutative algebra A is an homomorphism $\rho : R \to M_n(A)$ compatible with the trace map. If $A = \overline{\mathbf{k}}$ we think of this representation as a *geometric point*.

Remark 2.2.3. We have necessarily $n = d$, since

$$d = \rho(\mathrm{tr}(1)) = \mathrm{tr}(\rho(1)) = \mathrm{tr}(I) = n,$$

where I is the identity matrix of $M_n(A)$.

Before stating the main theorem of this section, we will give some examples of algebra with trace. In order to simplify the treatment and stick to a geometric language we assume, from now until the end of the chapter, that $\mathbf{k}$ is algebraically closed and of characteristic 0.

Example 2.2.4. Consider A to be an order in a finite dimensional central simple algebra D. This means that the center of A is a domain, A is torsion free over Z and, we have $D = A \otimes_Z Q(Z)$, where $Q(Z)$ is the field of fractions of Z. In other words, A embeds naturally in D which is its ring of fractions. If $\overline{Q(Z)}$ is the algebraic closure of $Q(Z)$, we have that $A \otimes_Z \overline{Q(Z)}$ is the full ring $M_d(\overline{Q(Z)})$. Hence we have on D, and on A, the usual reduced trace map $\mathrm{tr} : D \to Q(Z)$. It is well known that $tr(A) = Z$, if A is a finite Z module, Z is integrally closed and the characteristic is 0. So under this hypotheses A is an algebra of degree d. For more details cf. [Pro73] or [MR87].

Example 2.2.5. The second example is given by Azumaya algebras (cf [Art69]). Recall that:

Definition 2.2.6. An algebra R over a commutative ring A is called an *Azumaya algebra* of degree d over A, if there exists a faithfully flat extension B of A such that $R \otimes_A B$ is isomorphic to the algebra $M_d(B)$.

In this case it's easy to show that the ordinary trace maps R into A.

Let R be a finitely generated algebra, we want to describe it's d dimensional representation.

Theorem 2.2.7. *Assumes that $R \in \mathcal{C}_d$ is a finitely generated algebra. Set $T = \mathrm{tr}(R)$.*

(i) T is a finitely generated algebra, and R is s finite module over T. In particular T is the coordinate ring of an affine algebraic variety $X_T = \mathrm{Maxspec}\,(T)$.

(ii) The points of X_T parameterize equivalence classes of d-dimensional, trace compatible, and semisimple representations of R

(iii) Set $\mathrm{Spec}(R)$ equivalences classes of irreducible representations of R. The canonical map $\mathrm{Spec}(R) \xrightarrow{\chi} X_T$, induced by the central characters, is surjective and each fiber consists of all those irreducible representations of R which are irreducible components of the corresponding semisimple representation. In particular each irreducible representation of R has dimension at most d.

(iv) The set

$$\Omega_R = \{a \in Spec(T), \textit{such that the corresponding semisimple} \\ \textit{representation is irreducible}\}$$

is a Zariski open set. This is exactly the part of Spec(T) over which R is an Azumaya algebra of degree d.

Proof. See [DCP93] theorem 4.5, page 48. □

Remark 2.2.8. (1) If R is an order in a central simple algebra of degree d then T equals the center of R, furthermore since the central simple algebra splits in a d dimensional matrix algebra which one may consider as a generic irreducible representation, it is easily seen that the open set Ω_R is non empty.

(2) If T is a finitely generated module over a subalgebra Z_0, we can consider the finite surjective morphism

$$\tau : \operatorname{Spec}(T) \to \operatorname{Spec}(Z_0).$$

Then by the properness of τ we get that the set

$$\Omega_R^0 := \left\{a \in \operatorname{Spec}(Z_0) : \tau^{-1}(a) \subset \Omega_R\right\}$$

is a Zariski open dense subset of $\operatorname{Spec}(Z_0)$.

We will use this remark in the theory of quantum groups where there is a natural subalgebra Z_0 which appears in the picture.

3. TWISTED POLYNOMIAL ALGEBRAS

In this chapter we introduce the main notion of quasi polynomial algebras, or skew polynomial. Note that as the quantum enveloping algebras are the "quantum" version of the universal enveloping algebras of a Lie algebra, we can think that twisted polynomial algebras are the "quantum" version of the symmetric algebra of a Lie algebra. More details on twisted polynomial algebras can be found, for examples, in [DCP93] or [Man91].

3.1 *Useful notation and first properties*

Before giving the definition of twisted polynomial algebra, we want to introduce some notations, all will be useful in the sequel.

Let fix an invertible element $q \in \mathbb{C}$ different from 1 and -1 so that the fraction $\frac{1}{q-q^{-1}}$ is well defined. For all $n \in \mathbb{Z}$, set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} = q^{n-1} + q^{n-3} + \cdots + q^{-n+3} + q^{-n+1}.$$

We have the following relation:

$$\begin{array}{rcl} [-n] & = & -[n] \\ [n+m] & = & q^n [m] + q^m [n] \end{array}$$

Observe that if q is not a root of unity then $\forall\ n \in \mathbb{Z}$, non zero, $[n] \neq 0$. If q is a primitive l^{th} root of unity, with $l > 2$, define

$$e = \begin{cases} l \text{ if } l \text{ is odd} \\ \frac{l}{2} \text{ if } l \text{ is even.} \end{cases} .$$

Now is easy to check that

Property. *If q is a primitive l root of unity then*

(i) $[n] = 0 \Leftrightarrow n \equiv 0 \mod e$

(ii) $[n]^l = [n]$.

We can now define the q analogue of the factorials and of the binomial coefficients

Definition 3.1.1. For integer $0 \leq k \leq n$, set $[0]! = 1$,

$$[k]! = [1] \cdots [k] \,,$$

if $k > 0$, and

$$\left[\begin{array}{c} n \\ k \end{array} \right] = \frac{[n]!}{[k]!\,[n-k]!}.$$

Proposition 3.1.2. *If x and y are variables subject to the following relation $xy = q^2 yx$ then, for $n > 0$,*

$$(x+y)^n = \sum_{k=0}^{n} q^{-k(n-k)} \left[\begin{array}{c} n \\ k \end{array} \right] x^k y^{n-k}. \tag{3.1}$$

Proof. We begin by stating the q analogue of the Pascal identity:

$$q^k \left[\begin{array}{c} n+1 \\ k \end{array} \right] = q^{n+1} \left[\begin{array}{c} n+1 \\ k-1 \end{array} \right] + \left[\begin{array}{c} n \\ k \end{array} \right]$$

then by induction on n the statement follow □

Corollary 3.1.3. *If q is a primitive l root of unity,and $xy = q^2 yx$ then*

$$(x+y)^e = x^e + y^e.$$

Proof. Observe that

$$\left[\begin{array}{c} e \\ k \end{array} \right] = 0 \text{ for all } k \text{ such that } 0 < k < e.$$

Apply this in the formula 3.1 and the statement fallow. □

We give now some notations that will be useful in chapter 4 in order to define the relations of the quantum groups. Fix $d \in \mathbb{N}$, for all $n \in \mathbb{Z}$, set

$$[n]_d = \frac{q^n - q^{-n}}{q^d - q^{-d}}.$$

We can now extend the definitions of q-factorial and q-binomial, in the following way

Definition 3.1.4. For integer $0 \leq k \leq n$, set $[0]!_d = 1$,

$$[k]!_d = [1]_d \cdots [k]_d \,,$$

if $k > 0$, and

$$\left[\begin{array}{c} n \\ k \end{array} \right]_d = \frac{[n]!_d}{[k]!_d\,[n-k]!_d}.$$

3.2 Definition

Let A be an algebra over an algebraic closed field $\mathbf{k}$, and σ an automorphism of A.

Definition 3.2.1. A *twisted derivation* of A relative to σ is a linear map $D : A \to A$ such that:

$$D(ab) = D(a)b + \sigma(a)D(b)$$

$\forall\ a,\ b \in A$.

Definition 3.2.2. A twisted derivation D is called *inner*, if it exists in an element $a \in A$ such that:

$$D(b) = ab - \sigma(b)a$$

and we denote it $ad_\sigma a$.

Property. *Let $a \in A$ and σ be an automorphism such that $\sigma(a) = q^2 a$ where q is a scalar. Then*

$$(ad_\sigma a)^m (b) = \sum_{j=0}^{m} (-1)^j q^{j(m-1)} \left[\begin{array}{c} m \\ j \end{array} \right] a^{m-j} \sigma^j(b) a^j$$

Corollary 3.2.3. *Under the hypothesis of Property 3.2 we have:*

$$(ad_\sigma a)^e (x) = a^e x - \sigma^e(x) a^e$$

if q is a primitive l-th root of 1.

Fix an automorphism σ of A and a twisted derivation D of A relative to σ

Definition 3.2.4. We define the *twisted derivation algebra* $A_{\sigma,D}[x]$ in the indeterminate x to be the $\mathbf{k}$-module $A \otimes_{\mathbf{k}} \mathbf{k}[x]$ thought as formal polynomials with multiplications defined by the rule:

$$xa = \sigma(a)x + D(a).$$

When $D = 0$, we will call it *twisted polynomial algebra* and we denote it by $A_\sigma[x]$.

Let us notice that if $a,\ b \in A$ and a is invertible we can perform the change of variables

$$y := ax + b$$

and we see that

$$A_{\sigma,D}[x] = A_{\sigma',D'}[x],$$

for a suitable pair (σ', D'). In order to see this, it is better to make the formulas explicit separately when $b = 0$ and when $a = 1$. In the first case

$$yc = axc = a(\sigma(c)x + D(c)) = a(\sigma(c))a^{-1}y + aD(c),$$

and we see that the new automorphism σ' is the composition $\mathrm{Ad}(a)\sigma$, and $D' = aD$, where $\mathrm{Ad}(a)(x) = axa^{-1}$.

In the case $a = 1$, we have

$$yc = (x + b)c = \sigma(c)x + D(c) + bc = \sigma(c)y + D(c) + bc - \sigma(c)b,$$

so that $\sigma' = \sigma$ and $D' = D + \mathrm{ad}_\sigma b$. Summarizing we have

Proposition 3.2.5. *Changing σ, D to $\mathrm{Ad}(a)\sigma$, aD (resp. to σ, $D + \mathrm{ad}_\sigma b$) does not change the twisted derivation algebra up to isomorphism.*

Remark 3.2.6. It is clear that if A has no zero divisors, then the algebra $A_{\sigma,D}[x]$ and $A_\sigma[x, x^{-1}]$ also have no zero divisors.

Given a twisted polynomial algebra $A_{\sigma,D}[x]$, we can construct a natural filtration given by

$$\deg(p(x)) = n$$

where $p(x) = a_n x^n + \ldots + a_0$, $a_n \neq 0$. The associated graded algebra is clearly $A_\sigma[x]$.

Definition 3.2.7. We shall say that the algebra $A_\sigma[x]$ is a *simple degeneration* of $A_{\sigma,D}[x]$.

Example 3.2.8. Let A be an algebra over a field $\mathbf{k}$ of characteristic 0, let $x_1, \ldots, x_n$ be a set of generators of A and let Z_0 be a central subalgebra of A.

For each $i = 1, \ldots, n$, denote by A^i the subalgebra of A generated by $x_1, \ldots, x_i$, and let $Z_0^i = Z_0 \cap A^i$. We assume the following three conditions hold for each $i = 1, \ldots, n$:

1. $x_i x_j = b_{ij} x_j x_i + P_{ij}$ if $i > j$, where $b_{ij} \in \mathbf{k}$, $P_{ij} \in A^{i-1}$,
2. $\sigma_i(x_j) = b_{ij} x_j$ for $j < i$ define an automorphism of A^{i-1}.
3. $D_i(x_j) = P_{ij}$ for $j < i$.

We obtain

$$A^i = A^{i-1}_{\sigma_i, D_i}[x_i]$$

as twisted polynomial algebra. For every i, we may consider the twisted polynomial algebra $\overline{A}^i$ defined by the relation $x_i x_j = b_{ij} x_j x_i$ for $j < i$. We call this the *associated quasi polynomial algebra*.

Theorem 3.2.9. *Under the above assumptions, the quasi polynomial algebra $\overline{A} = \overline{A}^n$ is obtained from A by a sequence of simple degenerations.*

Proof. See [DCP93] theorem 5.3, page 56. □

3.3 Representation theory of twisted derivation algebras

We want to analyze some interesting cases of the previous constructions for which the resulting algebras are finite modules over their centers and thus we can develop for them the notion of degree and a good representation theory.

Let us first make a reduction, consider a finite dimensional semisimple algebra A over an algebraic closed field $\mathbf{k}$, let $\bigoplus_i \mathbf{k}e_i$ be the fixed points of the center of A under σ where the e_i are the central idempotents. We have

$$D(e_i) = D(e_i^2) = 2D(e_i)e_i,$$

hence $D(e_i) = 0$. It follows that, decomposing $A = \oplus_i Ae_i$, each component Ae_i is stable under σ and D and thus we have

$$A_{\sigma,D}[x] = \bigoplus_i (Ae_i)_{\sigma,D}[x].$$

This allows us to restrict our analysis to the case in which 1 is the only central idempotent.

The second reduction is described by the following:

Lemma 3.3.1. *Consider the algebra $A = \boldsymbol{k}^{\oplus n}$ with σ the cyclic permutation of the summands and let D be a twisted derivation of this algebra relative to σ. Then D is an inner twisted derivation.*

Proposition 3.3.2. *Let σ be the cyclic permutation of the summand of the algebra $\boldsymbol{k}^{\oplus n}$. Then*

(i) $\boldsymbol{k}_\sigma^{\oplus n}\left[x, x^{-1}\right]$ *is an Azumaya algebra of degree k over its center* $\boldsymbol{k}\left[x^n, x^{-n}\right]$.

(ii) $\boldsymbol{k}_\sigma^{\oplus n}\left[x, x^{-1}\right] \otimes_{\boldsymbol{k}[x^n, x^{-n}]} \boldsymbol{k}\left[x, x^{-1}\right]$ *is the algebra of $n \times n$ matrices over* $\boldsymbol{k}\left[x, x^{-1}\right]$.

Proof. [DCP93] proposition 6.1, page 56. □

Assume now that A is semisimple and that σ induces a cyclic permutation of the central idempotents

Lemma 3.3.3. *(i)* $A = M_d(\boldsymbol{k})^{\oplus n}$.

(ii) Let D be a twisted derivation of A relative to σ. Then the pair (σ, D) is equivalent to the pair $(\sigma', 0)$ where

$$\sigma'(a_1, a_2, \ldots, a_n) = (a_n, a_1, \ldots, a_{n-1})$$

Proof. [DCP93] lemma 6.2, page 57. □

Summarizing we have

Proposition 3.3.4. *Let A be a finite dimensional semisimple algebra over an algebraic closed field $\boldsymbol{k}$. Let σ be an automorphism of A which induces a cyclic permutation pf order n of the central idempotents of A. Let D be a twisted derivation of A relative to σ. Then:*

$$\begin{aligned} A_{\sigma,D}[x] &\cong M_d(\boldsymbol{k}) \otimes \boldsymbol{k}_\sigma^{\oplus n}[x], \\ A_{\sigma,D}[x, x^{-1}] &\cong M_d(\boldsymbol{k}) \otimes \boldsymbol{k}_\sigma^{\oplus n}[x, x^{-1}]. \end{aligned}$$

This last algebra is Azumaya of degree dk.

We can now globalize the previous construction. Let A be a prime algebra (i.e $aAb = 0$, a, $b \in A$, implies that $a = 0$ or $b = 0$) over the field $\mathbf{k}$ and let Z be the center of A. Then Z is a domain and A is torsion free over Z. Assume that A is a finite module over Z. Then A embeds in a finite dimensional central simple algebra $Q(A) = A \otimes_Z Q(Z)$, where $Q(Z)$ is the ring of fraction of Z. If $\overline{Q(Z)}$ denotes the algebraic closure of $Q(Z)$ we have that $A \otimes_Z \overline{Q(Z)}$ is the full ring $M_d(\overline{Q(Z)})$. Then d is called the degree of A.

Let σ be an automorphism of the algebra A and let D be a twisted derivation of A relative to σ. Assume that

(a) There is a subalgebra Z_0 of Z such that Z is finite over Z_0.

(b) D vanishes on Z_0 and σ restricted to Z_0 is the identity.

These assumptions imply that σ restricted to Z is an automorphism of finite order. Let d be the degree of A and let k be the order of σ on the center Z.

Definition 3.3.5. If A is an order in a finite dimensional central simple algebra and (σ, D) satisfy the previous conditions we shall say that the triple (A, σ, D) is *finite*.

Assume that the field $\mathbf{k}$ has characteristic 0. Then

Theorem 3.3.6. *Under the above assumptions the twisted polynomial algebra $A_{\sigma,D}[x]$ is an order in a central simple algebra of degree kd.*

Proof. [DCP93] theorem 6.3, page 58. □

Corollary 3.3.7. *Under the above assumptions, $A_{\sigma,D}[x]$ and $A_\sigma[x]$ have the same degree.*

Let A a prime algebra over a field $\mathbf{k}$ of characteristic 0, let $x_1, \dots, x_n$ be a set of generators of A and let Z_0 a central subalgebra of A. For each $i = 1, \dots, k$, denote by A^i the subalgebra of A generated by $x_1, \dots, x_k$, and let $Z_0^i = Z_0 \cap A^i$.

We assume, as in example 3.2.8, that the following three conditions hold for each $i = 1, \dots, k$:

(a) $x_i x_j = b_{ij} x_j x_i + P_{ij}$ if $i > j$, where $b_{ij} \in \mathbf{k}$, $P_{ij} \in A^{i-1}$.

(b) A^i is a finite module over Z_0^i.

(c) Formulas $\sigma_i(x_j) = b_{ij} x_j$ for $j < i$ define an automorphism of A^{i-1} which is the identity on Z_0^{i-1}.

Note that letting $D_i(x_j) = P_{ij}$ for $j < i$, we obtain $A^i = A^{i-1}_{\sigma,D}[x_i]$, so that A is an iterated twisted polynomial algebra. We may consider the twisted polynomial algebra $\overline{A}^i$ with zero derivations, so that the relations are $x_i x_j = b_{ij} x_j x_i$ for $j < i$. we call this the *associated twisted polynomial algebra.*

We can state the main theorem of this section.

Theorem 3.3.8. *Under the above assumptions, the degree of A is equal to the degree of the associated quasi polynomial algebra $\overline{A}$.*

Proof. By theorem 3.2.9 $\overline{A}$ is obtained from A with a sequence of simple degenerations, hence by corollary 3.3.7, it follows that they have the same degree. □

3.4 Representation theory of twisted polynomial algebras

Let $\mathbf{k}$ a field and $0 \neq q \in \mathbf{k}$ a given element. Given $n \times n$ skew symmetric matrix $H = (h_{ij})$ over $\mathbb{Z}$, we construct the *twisted polynomial algebra* $\mathbf{k}_H[x_1, \ldots, x_n]$. This is the algebra on generators $x_1, \ldots, x_n$ and the following defining relations:

$$x_i x_j = q^{h_{ij}} x_j x_i \tag{3.2}$$

for $i, j = 1, \ldots, n$. It can be viewed as an iterated twisted polynomial algebra with respect to any order of the x_i's. Similarly we can define the twisted Laurent polynomial algebra $\mathbf{k}_H[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$. Note that both algebras have no zero divisors.

To study its spectrum we start with a simple general lemma.

Lemma 3.4.1. *If M is an irreducible $A_\sigma[x]$ module, then there are two possibilities:*

(i) $x = 0$, hence M is actually an A module.

(ii) x is invertible, hence M is actually an $A_\sigma[x, x^{-1}]$ module.

Proof. It is clear that $\operatorname{im} x$ and $\ker x$ are submodules of M. □

Corollary 3.4.2. *In any irreducible $\mathbf{k}_H[x_1, \ldots, x_n]$ module, each element x_i is either 0 or invertible.*

Given $a = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$, we shall write $x^a = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$. The torus $(\mathbf{k}^\times)^n$ acts by automorphisms of the algebra $\mathbf{k}_H[x_1, \ldots, x_n]$ and $\mathbf{k}_H[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ in the usual way, the monomial x^a being a weight vector of weight a. Consider the group G of inner automorphisms of the twisted Laurent polynomials generate by conjugation by the variables x_i. Clearly G induces a group of automorphisms of the twisted polynomial algebra which are by 3.2 in this torus of automorphisms. In fact one can formalize this as follows:

Let $\Gamma := \left\{\alpha x^a : \alpha \in \mathbf{k}^\times\right\}$ be the set of non zero monomials. Then Γ is a group, $\mathbf{k}^\times$ is a central subgroup and $\Gamma/\mathbf{k}^\times$ is free abelian, the homomorphism $\Gamma \to (\mathbf{k}^\times)^n$ given by considering the associated inner automorphisms has as kernel the monomials in the center.

Let ϵ be a primitive l^{th} root of 1 in $\mathbf{k}$ and now take $q = \epsilon$. We consider the matrix H as a matrix of a homomorphism $H : \mathbb{Z}^n \to (\mathbb{Z}/l\mathbb{Z})^n$, and we denote by K the kernel of H and by h the cardinality of the image of H.

Proposition 3.4.3. *(i) The elements x^a with $a = (a_1, \ldots, a_n) \in K \cap \mathbb{Z}_+^n$ (resp. $a \in K$) form a basis of the center of $\boldsymbol{k}_H[x_1, x_2, \ldots, x_n]$ (resp. $\boldsymbol{k}_H[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$).*

(ii) Let $a^{(1)}, a^{(2)}, \ldots, a^{(h)}$ be a set of representative of $\mathbb{Z}^n \mod K$.

Then the monomials $x^{a^{(1)}}, x^{a^{(2)}} \ldots, x^{a^{(h)}}$ form a basis of the algebra $\boldsymbol{k}_H[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ over its center.

(iii) $\deg \boldsymbol{k}_H[x_1, \ldots, x_n] = \deg \boldsymbol{k}_H[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] = \sqrt{h}$

Proof. Define a skew symmetric bilinear form on $\mathbb{Z}^n$ by letting for $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathbb{Z}^n$:

$$\langle a|b\rangle = \sum_{i,j=1}^{n} h_{ij} a_i b_j.$$

Then we have

$$x^a x^b = \epsilon^{\langle a|b\rangle} x^b x^a. \tag{3.3}$$

Since the center is invariant with respect to the action of $(\mathbf{k}^\times)^n$, it must have a basis of elements from the form x^a. This together with 3.3 implies (i).

(ii) follows from (i) and the fact that

$$x^a x^b = \epsilon^{c(a|b)} x^{a+b} \tag{3.4}$$

where $c(a, b) = \sum_{i>j} h_{ij} a_i b_j$.

(iii) follows from (ii). □

Remark 3.4.4. The center of twisted polynomial algebra is the ring of invariants of a torus acting on a polynomial ring hence is integrally closed, moreover the algebra is finite over its center hence these algebras are closed under trace and in fact from 3.4 one can easily deduce a formula for the trace

$$\operatorname{tr} x^a = 0 \text{ if } x^a \text{ is not in the center.}$$

3.5 Maximal order

We have already stressed the importance of orders in a simple algebra, an important special case is the notion of maximal order which in a non commutative case replace the notion of an integrally closed domain. First we summarize some results on maximal order, more details can be found in [MR87], and at the end of the section we give a relation between maximal orders and twisted polynomial algebras.

Note. Every twisted polynomial algebra of the form $\mathbf{k}_H[x_1, \ldots, x_n]$ is an order.

Given an order R in a central simple algebra D an element $a \in R$ is a non zero divisor in R if and only if it is invertible in D, such an element is called *regular element.* Given two orders R_1 and R_2 let us consider the following condition:

There exist regular elements $a, b \in R_1$ such that $R_1 \subset aR_2b$. This relation generates an equivalence of orders and a *maximal order* is one which is maximal with respect to this equivalence.

Definition 3.5.1. An order R in a central simple algebra D is called *maximal order* if given any central element $c \in R$and an algebra S with $R \subset S \subset \frac{1}{c}R$ we have necessarily that $R = S$.

We remark an important property of maximal orders:

Property. *(i) The center Z of a maximal order R is integrally closed.*

(ii) If R is finitely generated algebra over a field $\mathbf{k}$ then R is a finite module over Z.

Corollary 3.5.2. *A maximal order in a center simple algebra D of degree d is closed under the reduced trace and hence it is an algebra in the category $\mathcal{C}_d$.*

Proof. See [DCP93], §4. □

We want to discuss now some criteria under which, by degeneration arguments, we can deduce that an algebra is a maximal order.

The setting that we have chosen is suggested by the work on quantum groups. We assume to have an algebra R, over some field $\mathbf{k}$, with a commutative subalgebra A and elements $x_1, \ldots, x_k$ satisfying some special conditions which we will presently explain.

Let us first introduce some notations. For an integral vector $\underline{n} := (n_1, \dots, n_k)$, $n_i \in \mathbb{N}$ we set $\deg \underline{n} := n_1 + \dots + n_k$, and $x^{\underline{n}} = x_1^{n_1} \dots x_k^{n_k}$ we call such element a *monomial*. Furthermore we define on the set of integral vectors the degree lexicographic ordering, i.e. set $\underline{n} < \underline{m}$ if either $\deg \underline{n} < \deg \underline{m}$ or $\deg \underline{n} = \deg \underline{m}$ but $\underline{n}$ is less than $\underline{m}$ in the usual lexicographic order, in this way $\mathbb{N}^k$ becomes an ordered monoid.

We now impose:

1. The monomials $x^{\underline{n}}$ are a basis of R as a left A module. Let us denote by
$$R_{\underline{n}} := \sum_{\underline{m} \leq \underline{n}} A x^{\underline{m}}.$$

2. The subspace $R_{\underline{n}}$ gives a structure of filtered algebra with respect to the ordered monoid $\mathbb{N}^k$. Furthermore we restrict the commutation relations among the elements x_i and A

3. $x_i x_j = a_{ij} x_j x_i + b_{ij}$ with $0 \neq a_{ij} \in \mathbf{k}$ and b_{ij} lower than $x_i x_j$ in the filtration.

4. $x_i a = \sigma_i(a) x_i +$ lower term with σ_i an automorphism of A. Notice that:

   (a) $x^{\underline{n}} x^{\underline{m}} = \lambda x^{\underline{m}} x^{\underline{n}} +$ lower term, with $0 \neq \lambda \in \mathbf{k}$.

   (b) The associated graded algebra $\overline{R}$ is a twisted polynomial ring over A. In fact the class $\overline{x}_i$ of the x_i satisfy
$$\begin{aligned} \overline{x}_i \overline{x}_j &= a_{ij} \overline{x}_j \overline{x}_i, \\ \overline{x}_i a &= \sigma_i(a) \overline{x}_i. \end{aligned}$$

5. A is integrally closed.

6. For every vector $\underline{n}$ there exists a monomial $a = x^{\underline{m}}$ such that $\underline{n} \leq \underline{m}$ and its class $\overline{a}$ is in the center of $\overline{R}$. Let us say for such an $\underline{m}$ is an *almost central* monomial. Such monomial have simple special commutation rules:
$$\begin{aligned} a x^{\underline{m}} &= x^{\underline{m}} a + \text{ lower term}, \forall a \in A. \\ x^{\underline{m}} x^{\underline{s}} &= x^{\underline{m}+\underline{s}} + \text{ lower term}, \forall \underline{s}. \end{aligned}$$

We can finally state our result:

Theorem 3.5.3. *Assume that R satisfies hypotheses $1-6$ then R is a maximal order.*

Proof. We follow De Concini and Kaç (cf. [DCP93] theorem 6.5, page 59 or [DCK90]). Let

$$z = bx^{\underline{r}} + \text{ lower term}$$

$b \in A$ be an element in the center of R and B be an algebra with $R \subset B \subset z^{-1}R$. We must show that $B = R$. Let us then take any element $u \in B$ and let $y := zu \in R$.

We develop $y := ax^{\underline{s}} + $ lower term, $a \in A$, we need to show that $u \in R$ by induction on $\underline{s}$. In order to do this we first want to prove that b divides a. Using hypotheses 6 and 4 we deduce that there is a monomial $v \in R$ such that $yv = z(uv)$ has the form

$$ax^{\underline{m}} + \text{ lower term}$$

with $x^{\underline{m}}$ an almost central monomial.

Next write $(yv)^t = z^{t-1}z(uv)^t$ and remark that $z(uv)^t \in R$. Now

$$(yv)^t = a^t x^{t\underline{m}} + \text{ lower term.}$$

Furthermore, we claim that

$$z^h = \lambda_h b^h x^{h\underline{r}} + \text{ lower term,}$$

for all h, $\lambda_h \in \mathbf{k}^*$. This is easily proved, since z is central, by induction, remarking that

$$z^h = z^{h-1}(bx^{\underline{r}} + \text{ lower term}) = bz^{h-1}x^{\underline{r}} + \text{ lower term}$$

We deduce that

$$\begin{aligned} a^t x^{t\underline{m}} + \text{ lower term} &= (yv)^t && (3.5a) \\ &= z^{t-1}z(uv)^t && (3.5b) \\ &= (\lambda_{t-1}b^{t-1}x^{(t-1)\underline{r}} + \text{lower term})z(uv)^t && (3.5c) \end{aligned}$$

This relation implies that for all t, b^{t-1} divides a^t in A, in other words $(a/b)^t \in b^{-1}A$. Since A is integrally closed (hypothesis 5), we deduce that b divides a in A as requested.

Next we claim that $\underline{r}$ divides $\underline{s}$. In fact from $y^t = z^{t-1}ku^t$ as for the identity 3.5, we deduce that in the monoid $\mathbb{N}^k$ the vector $(t-1)\underline{r}$ divides $t\underline{s}$ for all t, so $\underline{r}$ divides *underlines* and we can find an element w in R so that

$$x^{\underline{r}} = x^{\underline{s}}w + \text{ lower term.}$$

We can finish our argument by induction. Assume by contradiction that there is an element $u \in B$ and not in R we may choose it in such a way that the degree of $y = zu \in R$ is minimal. By the previous argument we know that $a = bf$, $f \in A$. Then $fzw = zfw$ has the same leading term as y and $u - fw \in B$. By induction $u - fw \in R$ which gives us a contradiction. □

Proposition 3.5.4. $\boldsymbol{k}_H[x_1, \ldots, x_n]$ *is a maximal order.*

Proof. All the hypotheses of theorem 3.5.3 are satisfied. □

Part II

QUANTUM GROUPS

4. GENERAL THEORY

We briefly recall the notations introduced in chapter 1. Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{h}$ a fixed Cartan subalgebra, and let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{b}$. We denote by $C = (a_{i,j})$ the Cartan matrix of $\mathfrak{g}$, so there exist d_i such that $(d_i a_{ij})$ is a positive symmetric matrix. Let R be the associated finite reduced root system, Λ its weight lattice and Q its roots lattice, $\mathcal{W}$ the Weyl group. The choice of $\mathfrak{b}$ gives us a set of positive root R^+, a set of simple roots $\Pi \subset R^+$ and a set of fundamental weights $w_1, \ldots, w_n \in \Lambda$.

4.1 Quantum universal enveloping Algebras

The quantum groups which will be the object of our study arise as q-analogues of the universal enveloping algebra of our semisimple Lie algebra $\mathfrak{g}$.

Definition 4.1.1. A *simply connected quantum group* $\mathcal{U}_q(\mathfrak{g})$ associated to the Cartan matrix C is an algebra over $\mathbb{C}(q)$ on generators E_i, F_i $(i = 1, \cdots, n)$, K_α $\alpha \in \Lambda$, subject to the following relations

$$\begin{cases} K_\alpha K_\beta = K_{\alpha+\beta} \\ K_0 = 1 \end{cases} \tag{4.1}$$

$$\begin{cases} \sigma_\alpha(E_i) = q^{(\alpha|\alpha_i)} E_i \\ \sigma_\alpha(F_i) = q^{-(\alpha|\alpha_i)} F_i \end{cases} \tag{4.2}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_{\alpha_i} - K_{-\alpha_i}}{q^{d_i} - q^{-d_i}} \tag{4.3}$$

$$\begin{cases} \sum\limits_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} E_i^{1-a_{ij}-s} E_j E_i^s = 0 \text{ if } i \neq j \\ \sum\limits_{s=0}^{1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^{1-a_{ij}-s} F_j F_i^s = 0 \text{ if } i \neq j. \end{cases} \tag{4.4}$$

Where $\begin{bmatrix} n \\ m \end{bmatrix}_{d_i}$ is the q binomial coefficient defined in section 3.1.

Note. When there is no possible confusion, we will simply denote $\mathcal{U}_q(\mathfrak{g})$ by $\mathcal{U}$.

Note. One should think of E_i and F_i as q-analogues of the Chevalley generators of $\mathfrak{g}$.

Theorem 4.1.2. *$\mathcal{U}$ has a Hopf algebra structure with comultiplication Δ, antipode S and counit η defined by:*

- $\begin{cases} \Delta(E_i) = E_i \otimes 1 + K_{\alpha_i} \otimes E_i \\ \Delta(F_i) = F_i \otimes K_{-\alpha_i} + 1 \otimes F_i \\ \Delta(K_\alpha) = K_\alpha \otimes K_\alpha \end{cases}$

- $\begin{cases} S(E_i) = -K_{\alpha_i} E_i \\ S(F_i) = -F_i K_{\alpha_i} \\ S(K_\alpha) = K_{-\alpha} \end{cases}$

- $\begin{cases} \eta(E_i) = 0 \\ \eta(F_i) = 0 \\ \eta(K_\alpha) = 1 \end{cases}$

Proof. See [Lus93]. □

Note. The quantum group in the sense of *Drinfel'd-Jimbo* is the subalgebra U_Q over $\mathbb{C}(q)$ generated by E_i, F_i, $K_i^{\pm 1} = K_{\pm\alpha_i}$ $(i = 1, \cdots, n)$, we call it also *adjoint quantum group*. More generally, for any lattice M between Λ and Q, we can define $\mathcal{U}_M$ to be the quantum group generated by the E_i, F_i $(i = 1, \cdots, n)$ and the K_β with $\beta \in M$.

We denote by $\mathcal{U}^+$, $\mathcal{U}^-$ and $\mathcal{U}^0$ the $\mathbb{C}(q)$-subalgebra of $\mathcal{U}_M$ generated by the E_i, the F_i and K_β respectively. The algebras $\mathcal{U}^+$ and $\mathcal{U}^-$ are not Hopf subalgebras as one immediately sees from theorem 4.1.2. On the other hand, the algebra $\mathcal{U}^{\geq 0} := \mathcal{U}^+\mathcal{U}^0$ and $\mathcal{U}^{\leq 0} := \mathcal{U}^0\mathcal{U}^-$ are Hopf subalgebras and we shall think to them as quantum deformation of the enveloping algebras $\mathcal{U}(\mathfrak{b})$ and $\mathcal{U}(\mathfrak{b}^-)$, we denote them $\mathcal{U}_q(\mathfrak{b})$ and $\mathcal{U}_q(\mathfrak{b}^-)$.

In fact we are interested in the study of a common generalization of $\mathcal{U}_q(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{b})$, namely $\mathcal{U}_q(\mathfrak{p})$ for $\mathfrak{b} \subseteq \mathfrak{p} \subseteq \mathfrak{g}$ a parabolic subalgebra.

4.1.1 P.B.W. basis

First, following Lusztig [Lus93], we define an action of the braid group $\mathcal{B}_{\mathcal{W}}$ (associated to $\mathcal{W}$). Denote by T_i the canonical generators of $\mathcal{B}_{\mathcal{W}}$, we define the action as an automorphism of $\mathcal{U}$, by the formulas:

$$T_i K_\lambda = K_{s_i(\lambda)} \tag{4.5a}$$

$$T_i E_i = -F_i K_i \tag{4.5b}$$

$$T_i F_i = -K_i^{-1} E_i \tag{4.5c}$$

$$T_i E_j = \left(-adE_i^{(-a_{i,j})}\right)(E_j) \tag{4.5d}$$

where if $\Delta(x) = \sum x_j \otimes y_j$, then $ad(x)(y) = \sum x_j y S(y_j)$.

Now we use the braid group to construct analogues of the root vectors associated to non simple roots.

Let us take a reduced expression $\omega_0 = s_{i_1} \dots s_{i_N}$ for the longest element in the Weyl group $\mathcal{W}$. Setting $\beta_j = s_{i_1} \cdots s_{i_{j-1}}(\alpha_j)$, we get a total order on the set of positive root. We define the elements $E_{\beta_j} = T_{i_1} \dots T_{i_{j-1}}(E_{i_j})$ and $F_{\beta_j} = T_{i_1} \dots T_{i_{j-1}}(F_{i_j})$. Note that this elements depend on the choice of the reduced expression.

Lemma 4.1.3. *(i)* $E_{\beta_j} \in \mathcal{U}^+$, $\forall\ i = 1 \dots N$

(ii) $F_{\beta_j} \in \mathcal{U}^-$, $\forall\ i = 1 \dots N$

Lemma 4.1.4. *(i) The monomials* $E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N}$ *are a* $\mathbb{C}(q)$ *basis of* $\mathcal{U}^+$

(ii) The monomials $F_{\beta_1}^{k_1} \cdots F_{\beta_N}^{k_N}$ *are a* $\mathbb{C}(q)$ *basis of* $\mathcal{U}^-$

Poincaré Birkhoff Witt Theorem. *The monomials*

$$E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N} K_\alpha F_{\beta_N}^{k_N} \cdots F_{\beta_1}^{k_1}$$

are a $\mathbb{C}(q)$ *basis of* $\mathcal{U}$*. In fact as vector spaces, we have the tensor product decomposition,*

$$\mathcal{U} = \mathcal{U}^+ \otimes \mathcal{U}^0 \otimes \mathcal{U}^-$$

Proof. See [Lus93]. □

Levendorskii Soibelman relation. *For* $i < j$ *one has*

(i)

$$E_{\beta_j} E_{\beta_i} - q^{(\beta_i|\beta_j)} E_{\beta_i} E_{\beta_j} = \sum_{k \in \mathbb{Z}_+^N} c_k E^k \tag{4.6}$$

where $c_k \in \mathbb{C}[q, q^{-1}]$ *and* $c_k \neq 0$ *only when* $k = (k_1, \dots, k_N)$ *is such that* $k_s = 0$ *for* $s \leq i$ *and* $s \geq j$*, and* $E^k = E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N}$.

(ii)

$$F_{\beta_j} F_{\beta_i} - q^{-(\beta_i|\beta_j)} F_{\beta_i} F_{\beta_j} = \sum_{k \in \mathbb{Z}_+^N} c_k F^k \tag{4.7}$$

where $c_k \in \mathbb{C}[q, q^{-1}]$ *and* $c_k \neq 0$ *only when* $k = (k_1, \cdots, k_N)$ *is such that* $k_s = 0$ *for* $s \leq i$ *and* $s \geq j$*, and* $F^k = F_{\beta_N}^{k_N} \cdots F_{\beta_1}^{k_1}$.

Proof. See [LS91b]. □

An immediate corollary is the following: Let $\omega \in \mathcal{W}$. Choose a reduce expression for it, $\omega = s_{i_1} \dots s_{i_k}$, which we complete to a reduced expression $\omega_0 = s_{i_1} \dots s_{i_N}$ of the longest element of $\mathcal{W}$. Consider the elements E_{β_j}, $j = 1, \dots, k$. Then we have:

Proposition 4.1.5. *(i) The elements E_{β_j}, $j = 1, \dots, k$, generated a subalgebra $\mathcal{U}^\omega$ which is independent of the choice of the reduced expression of ω.*

(ii) If $\omega' = ws$ with s a simple reflection and $l(\omega') = l(\omega) + 1 = k + 1$. then $\mathcal{U}^{\omega'}$ is a twisted polynomial algebra of type $\mathcal{U}^\omega_{\sigma,D}\left[E_{\beta_{k+1}}\right]$, where σ and D are given by the following formula, given in 4.6

$$\sigma\left(E_{\beta_j}\right) = q^{(\beta_j|\beta_{k+1})} E_{\beta_j} \tag{4.8a}$$

$$D\left(E_{\beta_j}\right) = \sum_{k \in \mathbb{Z}_+^N} c_k E^k. \tag{4.8b}$$

Proof. See [DCP93]. □

The elements K_α clearly normalize the algebra $\mathcal{U}^\omega$ and when we add them to these algebras we are performing an iterated construction of Laurent twisted polynomials. The related algebras will be called B^ω.

4.1.2 Degenerations of quantum groups

We want to construct some degenerations of our algebra $\mathcal{U}_q(\mathfrak{g})$ as the graded algebra associated to suitable filtration.

Definition 4.1.6. Consider the monomials $M_{k,r,\alpha} = F^k K_\alpha E^r$, where $k = (k_1, \dots, k_N)$, $r = (r_1, \dots, r_N) \in \mathbb{Z}_+^N$ and $\alpha \in \Lambda$. The *total height* of $M_{k,r,\alpha} = F^k K_\alpha E^r$ is defined by

$$d_0(M_{k,r,\alpha} = F^k K_\alpha E^r) = \sum_i (k_i + r_i) \operatorname{ht} \beta_i,$$

And its *total degree* by

$$d(M_{k,r,\alpha}) = (k_N, \dots, k_1, r_1, \dots, r_N, d_0) \in \mathbb{Z}_+^{2N+1},$$

where, $\operatorname{ht}\beta$ is the usual height of a root with respect to our choice of simple roots.

We shall view $\mathbb{Z}_+^{2N+1}$ as a total ordered semigroup with the lexicographic order $<$. L.S. relations allow us to introduce a $\mathbb{Z}_+^{2N+1}$-filtration of the algebra $\mathcal{U}$ by letting $\mathcal{U}_s$, $s \in \mathbb{Z}_+^{2N+1}$ be the span of the monomials $M_{k,r,\alpha}$ such that $d(M_{k,r,\alpha}) \leq s$.

Proposition 4.1.7. *The associated graded algebra $\operatorname{Gr}\mathcal{U}$ of the $\mathbb{Z}_+^{2N+1}$-filtered algebra $\mathcal{U}$ is an algebra over $\mathbb{C}(q)$, on generators $\mathcal{E}_\alpha$, $\alpha \in R$ and $\mathcal{K}_\beta$, $\beta \in \Lambda$,* *subject to the following relations:*

$$\mathcal{K}_\alpha \mathcal{K}_\beta = \mathcal{K}_{\alpha+\beta},\ \mathcal{K}_0 = 1; \tag{4.9a}$$

$$\mathcal{K}_\alpha \mathcal{E}_\beta = q^{(\alpha|\beta)} \mathcal{E}_\beta \mathcal{K}_\alpha; \tag{4.9b}$$

$$\mathcal{E}_\alpha \mathcal{E}_{-\beta} = \mathcal{E}_{-\beta} \mathcal{E}_\alpha \ \textit{if}\ \alpha, \beta \in R^+ \tag{4.9c}$$

$$\left\{ \begin{array}{l} \mathcal{E}_\alpha \mathcal{E}_\beta = q^{(\alpha|\beta)} \mathcal{E}_\beta \mathcal{E}_\alpha \\ \mathcal{E}_{-\alpha} \mathcal{E}_{-\beta} = q^{(\alpha|\beta)} \mathcal{E}_{-\beta} \mathcal{E}_{-\alpha} \end{array} \right. \quad \textit{if}\ \alpha, \beta \in R^+ \ \textit{and}\ \alpha > \beta. \tag{4.9d}$$

Proof. See [DCP93], §10. □

Remark 4.1.8. *a*) Considering the degree by total height d_0, we obtain a $\mathbb{Z}_+$-filtration of $\mathcal{U}$, let $\mathcal{U}^{(0)} = \mathrm{Gr}\mathcal{U}$ the associated graded algebra. We define by induction $\mathcal{U}^{(i)}$ the graded algebra associated to $\mathcal{U}^{(i-1)}$ with respect to the $\mathbb{Z}_+$-filtration given by

$$d_i\left(M_{k,r,\alpha}\right) = \left\{ \begin{array}{l} r_{N-i+1} \text{ if } 1 \leq i \leq N \\ k_{i-N} \text{ if } N+1 \leq i \leq 2N \end{array} \right.$$

It is clear that at last step we get the algebra $\mathrm{Gr}\mathcal{U}$ defined by 4.9, i.e.

$$\mathcal{U}^{(2N)} \cong \mathrm{Gr}\mathcal{U}$$

b) The algebra $\mathrm{Gr}\mathcal{U}$ is a twisted polynomial algebra over $\mathbb{C}(q)$ on generators $\mathcal{E}_{\beta_1}, \ldots, \mathcal{E}_{\beta_N}, \mathcal{E}_{-\beta_N}, \ldots \mathcal{E}_{-\beta_1}$, and $\mathcal{K}_1, \ldots, \mathcal{K}_n$, with the element $\mathcal{K}_i$ inverted.

A first application of this methods is:

Theorem 4.1.9. *The algebra $\mathcal{U}$ has no zero divisors.*

Proof. Follows from remark 3.2.6. □

4.2 Quantum groups at root of unity

To obtain from $\mathcal{U}$ a well defined Hopf algebra by specializing q to an arbitrary non zero complex number ϵ, one can construct an *integral form* of $\mathcal{U}$.

Definition 4.2.1. An *integral form* $\mathcal{U}_\mathcal{A}$ is a $\mathcal{A}$ subalgebra, where $\mathcal{A} = \mathbb{C}[q, q^{-1}]$, such that the natural map

$$\mathcal{U}_\mathcal{A} \otimes_\mathcal{A} \mathbb{C}(q) \mapsto \mathcal{U}$$

is an isomorphism of $\mathbb{C}(q)$ algebra. We define

$$\mathcal{U}_\epsilon = \mathcal{U}_\mathcal{A} \otimes_\mathcal{A} \mathbb{C}$$

using the homomorphism $\mathcal{A} \mapsto \mathbb{C}$ taking q to ϵ.

There are two different candidates for $\mathcal{U}_{\mathcal{A}}$ the non restricted and the restricted integral form, which lead to different specializations (with markedly different representation theories) for certain values of ϵ. We are interested in the non restricted form. For more details one can see [CP95].

Introduce the elements

$$[K_i; m]_{q_i} = \frac{K_i q_i^m - K_i^{-1} q_i^{-m}}{q_i - q_i^{-1}} \in \mathcal{U}^0$$

with $m \geq 0$, where $q_i = q^{d_i}$.

Definition 4.2.2. The algebra $\mathcal{U}_{\mathcal{A}}$ is the $\mathcal{A}$ subalgebra of $\mathcal{U}$ generated by the elements E_i, F_i, $K_i^{\pm 1}$ and $L_i = [K_i; 0]_{q_i}$, for $i = 1, \dots, n$. With the map Δ, S and η defined on the first set of generators as in 4.1.2 and with

$$\Delta(L_i) = L_i \otimes K_i + K_i^{-1} \otimes L_i \tag{4.10a}$$

$$S(L_i) = -L_i \tag{4.10b}$$

$$\eta(L_i) = 0 \tag{4.10c}$$

Note. The defining relation of $\mathcal{U}_{\mathcal{A}}$ are as in 4.1.1 replacing 4.3 by

$$E_i F_j - F_j E_i = \delta_{ij} L_i$$

and adding the relation

$$(q_i - q_i^{-1}) L_i = K_i - K_i^{-1}$$

Proposition 4.2.3. *$\mathcal{U}_{\mathcal{A}}$ with the previous definition is a Hopf algebra. Moreover, $\mathcal{U}_{\mathcal{A}}$ is an integral form of $\mathcal{U}$.*

Proof. See [CP95] or [DCP93] §12. □

Proposition 4.2.4. *If $\epsilon^{2d_i} \neq 1$ for all i, then*

(i) $\mathcal{U}_\epsilon$ is generated over $\mathbb{C}$ by the elements E_i, F_i, and $K_i^{\pm 1}$ with defining relations obtained from those in 4.1.1 by replacing q by ϵ

(ii) The monomials

$$E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N} K_\alpha F_{\beta_N}^{k_N} \cdots F_{\beta_1}^{k_1}$$

are a $\mathbb{C}$ basis of $\mathcal{U}_\epsilon$.

(iii) The L.S. relations holds in $\mathcal{U}_\epsilon$.

Proof. See [DCP93] §12. □

4.2.1 The center of $\mathcal{U}_\epsilon$

The center of $\mathcal{U}_\epsilon$ consists of "two parts", one coming from the center at a "generic q", the other from the fact that ϵ is a root of 1. We begin by describing the center at "q generic".

Let Z be the center of $\mathcal{U}$. Any element $z \in Z$ can be written as a linear combinations of the elements of the basis of $\mathcal{U}$ given by the P.B.W. theorem. Since z commutes with the K_i for all i, it follows that

$$z = \sum_{\eta \in Q^+} \sum_{r,t \in Par(\eta)} E^r \varphi_{r,t} F^t$$

where $\varphi_{r,t} \in \mathcal{U}^0$ and

$$Par(\eta) = \left\{ (m_1, \ldots, m_N) \in \mathbb{N}^N : \sum_i m_i \alpha_i = \eta \right\}.$$

Definition 4.2.5. The map $h : Z \mapsto \mathcal{U}^0$ defined by $h(z) = \varphi_{0,0}$ is called the *Harish Chandra homomorphism*

Property. *h is a homomorphism of algebras.*

Proof. See [DCP93], §18. □

h allows us to describe Z as a ring. Any element $\phi \in \mathcal{U}^0$ may be regarded as a $\mathbb{C}(q)$-valued function on the weight lattice Λ in an obvious way: if $\phi = \prod_{i=1}^n K_i^{t_i}$, where $t_i \in \mathbb{Z}$ for all i, set $\forall\ \lambda \in \Lambda$

$$\phi(\lambda) = q^{\sum_i t_i (\alpha_i | \lambda)}$$

and extended to $\mathcal{U}^0$ by linearity. Define an automorphism of $\mathbb{C}(q)$-algebras $\gamma : \mathcal{U}^0 \mapsto \mathcal{U}^0$ by setting $\gamma(K_i) = q_i K_i$. Then

Property. *For all $\phi \in \mathcal{U}^0$*

$$\gamma(\phi)(\lambda) = \phi(\lambda + \rho)$$

where $\rho = \frac{1}{2}(\alpha_1 + \ldots + \alpha_n)$.

Let $Q_2^* = \{\sigma : Q \mapsto \mathbb{Z}_2\}$, it is easy to see that Q_2^* acts on $\mathcal{U}$ as a group of automorphisms by

$$\sigma \cdot K_\beta = \sigma(\beta) K_\beta \tag{4.11}$$

$$\sigma \cdot F_\alpha = \sigma(\alpha) F_\alpha \tag{4.12}$$

$$\sigma \cdot E_\alpha = E_\alpha \tag{4.13}$$

for all $\alpha \in R^+$, $\beta \in Q$, and $\sigma \in Q_2^*$. Note that $\mathcal{W}$ acts on Q_2^*:

$$(\omega \cdot \sigma)(\beta) = \sigma(\omega^{-1}(\beta));$$

moreover, the action of Q_2^* can be obviously extended to an action of the semidirect product $\mathcal{W} \ltimes Q_2^*$.

Let $\widetilde{\mathcal{W}}$ the subgroup generated by all conjugates $\sigma\mathcal{W}\sigma^{-1}$ of $\mathcal{W}$ by elements $\sigma \in Q_2^*$.

Theorem 4.2.6. *The homomorphism $\gamma \circ h : Z \mapsto \mathcal{U}^0$ is injective and its image is precisely the set $\mathcal{U}^{0\widetilde{W}}$ of fixed points of the action of $\widetilde{W}$ on $\mathcal{U}^0$.*

Proof. See [DCP93] or [CP95]. □

Example 4.2.7. Let $\mathfrak{g} = sl_2(\mathbb{C})$. Then $\widetilde{\mathcal{W}} = \mathcal{W}$ and $\mathcal{U}^{0\widetilde{\mathcal{W}}}$ consist of the Laurent polynomials in K_1 which are invariant under $K_1 \mapsto K_1^{-1}$. Thus $\mathcal{U}^{0\widetilde{\mathcal{W}}}$ is generated as an algebra over $\mathbb{C}(q)$ by

$$\phi = \frac{K_1 + K_{-1}^{-1}}{(q - q^{-1})^2}.$$

It easy to check that the *quantum Casimir element*

$$\Omega = \frac{qK_1 + q^{-1}K_{-1}^{-1}}{(q - q^{-1})^2} + EF$$

lies in Z and $\gamma^{-1}(h(\Omega)) = \phi$. It follows from theorem 4.2.6 that Ω generates Z as $\mathbb{C}(q)$-algebra.

We see now the quantum analogue of the Harish Chandra's theorem on the central characters of the classical universal enveloping algebra. Let $\lambda \in \Lambda$ and define a homomorphism $\lambda : \mathcal{U}^0 \mapsto \mathbb{C}(q)$ by $\mathcal{K}_i \mapsto q^{(\alpha_i|\lambda)}$. Let

$$\chi_{q,\lambda} = \lambda \circ \gamma^{-1} \circ h : Z \mapsto \mathbb{C}(q)$$

Theorem 4.2.8. *Let λ, $\mu \in \Lambda$. Then $\chi_\lambda = \chi\mu$ if and only if $\mu = \omega(\lambda)$ for some $\omega \in \mathcal{W}$.*

Proof. See [CP95] □

Suppose now that $\epsilon \in \mathbb{C}$ is a l^{th} root of the unity such that l is odd and $l > d_i$ for all i. Our aim is to describe the center Z_ϵ of $\mathcal{U}_\epsilon$. We shall assume a different approach from that used in the generic case. Let $Z_1 = (Z \cap \mathcal{A}) / (q - \epsilon)$, we call it the Harish Chandra part of the center. This is not the full center, in fact we have

Proposition 4.2.9. *The elements E_α^l, F_α^l for $\alpha \in R^+$, and K_i^l for $i = 1, \ldots, n$ lie in Z_ϵ.*

Proof. See [DCP93], §21. □

For $\alpha \in R^+$, $\beta \in Q$, set $e_\alpha = E_\alpha^l$, $f_\alpha = F_\alpha^l$ and $k_\beta = K_\beta^l$; we shall often write e_i and f_i for e_{α_i} and f_{α_i}. Let Z_0 (resp. Z_0^+, Z_0^-, and Z_0^0 be the subalgebra of Z_ϵ generated by the e_α, f_α and $k_i^\pm$ (resp.e_α, f_α and $k_i^\pm$).

Proposition 4.2.10. *(i) We have $Z_0^\pm \subset \mathcal{U}_\epsilon^\pm$.*

(ii) Multiplication defines an isomorphism of algebras

$$Z_0^- \otimes Z_0^0 \otimes Z_0^+ \mapsto Z_0.$$

(iii) Z_0^0 is the algebra of Laurent polynomial in the k_i, and $Z_0^\pm$ is the polynomial algebra with generators the e_α and f_α respectively.

(iv) We have $Z_0^\pm = \mathcal{U}_\epsilon^\pm \cap Z_\epsilon$

(v) The subalgebra Z_0 of Z_ϵ is preserved by the braid group algebra automorphism T_i.

(vi) $\mathcal{U}_\epsilon$ is a free Z_0 module with basis the set of monomial

$$E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N} K_1^{s_1} \dots K_n^{s_n} F_{\beta_N}^{t_N} \cdots F_{\beta_1}^{t_1}$$

for which $0 \leq t_j, s_i, k_j < l$, for $i = 1, \dots, n$ and $j = 1, \dots, N$.

Proof. See [DCP93], §21. □

Therefore, we can completely describe the center of U_ϵ.

Theorem 4.2.11. *Z_ϵ is generated by Z_1 and Z_0.*

Proof. See [DCP93], §21. □

The preceding proposition shows that $\mathcal{U}_\epsilon$ is a finite Z_0 module. It follows that $Z_\epsilon \subset \mathcal{U}_\epsilon$ is finite over Z_0, and hence integral over Z_0. By the Hilbert basis theorem, Z_ϵ is a finitely generated algebra. Thus, the affine schemes $\mathrm{Spec}(Z_\epsilon)$ and $\mathrm{Spec}(Z_0)$, namely the sets of algebra homomorphism from Z_ϵ and Z_0 to $\mathbb{C}$, are algebraic varieties. In fact, it is obvious that $\mathrm{Spec}(Z_0)$ is isomorphic to $\mathbb{C}^{2N} \times (\mathbb{C}^*)^n$. Moreover the inclusion $Z_0 \hookrightarrow Z_\epsilon$ induces a projection $\tau : \mathrm{Spec}(Z_\epsilon) \mapsto \mathrm{Spec}(Z_0)$, and we have

Proposition 4.2.12. $\mathrm{Spec}(Z_\epsilon)$ *is a normal affine variety and τ is a finite (surjective) map of degree l^n.*

Proof. See [CP95] or [DCP93] §21. □

We conclude this section by discussing the relation between the center and the Hopf algebra structure of $\mathcal{U}_\epsilon$.

Property. *We have*

$$\Delta(e_i) = e_i \otimes 1 + k_1^{-1} \otimes e_i$$
$$\Delta(f_i) = f_i \otimes k_i + 1 \otimes f_i$$
$$\Delta(k_i) = k_i \otimes k_i$$

Proposition 4.2.13. *Z_0 is a Hopf subalgebra of $\mathcal{U}_\epsilon$, as are Z_0^0, $Z_0^{\geq 0} = Z_0^0 Z_0^+$ and $Z_0^{\leq 0} = Z_0^- Z_0^0$.*

It follows that $\mathrm{Spec}(Z_0)$ inherits a Lie group structure from the Hopf structure of $\mathcal{U}_\epsilon$. In fact

Property. *The formula*

$$\{z, z'\} = \lim_{q \to \epsilon} \frac{zz' - z'z}{l\,(q^l - q^{-l})}$$

defines a Poisson bracket on Z_0 which gives to $\mathrm{Spec}\, Z_0$ the structure of a Poisson Lie group.

In order to study in more details the Poisson structure of Z_0, we must introduce some extra structure. For every $q \in \mathbb{C}$, define derivations e_i and f_i of $\mathcal{U}_q$ by

$$\underline{e}_i(u) = \left[\frac{E_i^l}{[l]_{q_i}!}, u\right], \tag{4.14}$$

$$\underline{f}_i(u) = \left[\frac{F_i^l}{[l]_{q_i}!}, u\right]. \tag{4.15}$$

Note that, if we specialize to $q = \epsilon$, we obtain at first sight an indeterminate result, since the E_i^l and F_i^l are central and $[l]_{\epsilon_i} = 0$. However, we have

Proposition 4.2.14. *On specializing to $q = \epsilon_i = \epsilon^{d_i}$ the formulas 4.14 induces well defined derivation of $\mathcal{U}_\epsilon$. In fact, we have the following explicit formulas:*

$$\underline{e}_i(E_j) = \frac{1}{2}\sum_{r=1}^{-a_{ij}} \left[\begin{array}{c} -a_{ij} \\ r \end{array}\right]_{\epsilon_i} \left(F_i^r F_j F_i^{l-r} - F_i^{l-r} F_j F_i^r\right),$$
$$\underline{e}_i(F_j) = \frac{1}{l}\delta_{i,j}(\epsilon_i - \epsilon_i^{-1})^{l-2}(K_i\epsilon_i - K_i^{-1}\epsilon_i^{-1})E_i^{l-1}$$
$$\underline{e}_i(K_j^{\pm 1}) = \mp\frac{1}{2l} a_{ij}(\epsilon_i - \epsilon_i^{-1})^l F_i^l K_j^{\pm 1}$$

and $\underline{f}_i$ is obtained from $\underline{e}_i$ by using

$$T_{\omega_0}\underline{e}_{\bar{i}}T_{\omega_0}^{-1} = \underline{f}_i$$

where $i \mapsto \bar{i}$ is the permutation of the nodes of the Dynkin diagram of $\mathfrak{g}$ such that $\omega_0(\alpha_i) = -\alpha_{\bar{i}}$.

Let G be the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$, let $H \subset G$ be the maximal torus of G where $LieH = \mathfrak{h}$, and let $N^{\pm}$ be the unipotent subgroups of G with Lie algebra $\mathfrak{n}^{\pm}$. Note that H is canonically identified with $\mathrm{Spec}(Z_0^0)$ by the paring

$$(\exp(\eta), k_i) = \exp\left(2\pi\sqrt{-1}\alpha_i(\eta)\right)$$

for $\eta \in \mathfrak{h}$, the Lie algebra of H. The product $G^0 = N^- H N^+$ is well known to be a dense open subset of G (in the complex topology), called the *big cell.*

We define maps

$$\mathbf{E} : \mathrm{Spec}(Z_0^+) \to N^+ \tag{4.16}$$
$$\mathbf{F} : \mathrm{Spec}(Z_0^-) \to N^- \tag{4.17}$$
$$\mathbf{K} : \mathrm{Spec}(Z_0^0) \to H \tag{4.18}$$

and by multiplication a map

$$\pi : \mathbf{EKF} : \mathrm{Spec}(Z_0) = \mathrm{Spec}(Z_0^+) \times \mathrm{Spec}(Z_0^0) \times \mathrm{Spec}(Z_0^-) \to G$$

as follows. Fix a reduced expression of the longest element of $\mathcal{W}$, $\omega_0 = s_{i_1} \dots s_{i_N}$, and let $\overline{E}_{\beta_1}, \dots, \overline{E}_{\beta_N}$ be the corresponding negative root of $\mathfrak{g}$. Let

$$f_{\beta_k} = \left(\epsilon_{i_k} - \epsilon_{i_k}^{-1}\right) T_{i_1} \dots T_{i_{k-1}}(f_{i_k}) \in Z_0$$

which we regard as a complex valued function on $\mathrm{Spec}(Z_0)$. Then we define maps $\mathbf{E}$, $\mathbf{F}$ and $\mathbf{K}$ to be the products

$$\begin{aligned} \mathbf{F} &= \exp\left(f_{\beta_N}\overline{F}_{\beta_N}\right) \dots \exp\left(f_{\beta_1}\overline{F}_{\beta_1}\right) \\ \mathbf{E} &= \exp\left(T_{\omega_0}(f_{\beta_N})T_{\omega_0}(\overline{F}_{\beta_N})\right) \dots \exp\left(T_{\omega_0}(f_{\beta_1})T_{\omega_0}(\overline{F}_{\beta_1})\right) \\ \mathbf{K}(h) &= h^2 \end{aligned}$$

where $h \in H$

Proposition 4.2.15. *The product map $\pi = \boldsymbol{FKE} : \mathrm{Spec}(Z_0) \to G$ is independent of the choice of reduced decomposition of ω_0, and is a covering of degree 2^n.*

Proof. See [DCP93], §16. □

Theorem 4.2.16. *(i) Consider the Poisson structure on G defined in example 1.2.14, then we have an identification of $\mathrm{Spec}(Z_0)$ with a Poisson dual to G. In particular Lie $\mathrm{Spec}(Z_0) = \mathfrak{s}$, where*

$$\mathfrak{s} = \left\{(x, y) \in \mathfrak{b}_+ \oplus \mathfrak{b}_- : x_{\mathfrak{h}} + y_{\mathfrak{h}} = 0\right\},$$

(ii) The symplectic leaves of $\mathrm{Spec}(Z_0)$ coincide with the preimages of the conjugacy classes in G under π.

(iii) If $C \subset G$ is a conjugacy class and $\dim C > 0$ then $\pi^{-1}(C)$ is connected.

Proof. see [DCKP92] or [DCP93] §16. □

4.3 Parametrization of irreducible representation of $\mathcal{U}_\epsilon$

As usual we assume that ϵ is a primitive l^{th} root of the unity with l odd and $l > d_i$ for all i. All representation are on complex vector space.

We know that Z_ϵ acts by scalar operators on V (cf [CP95]), so there exist an homomorphism $\chi_V : Z_\epsilon \mapsto \mathbb{C}$, the *central character of* V, such that

$$z \cdot v = \chi_V(z)v$$

for all $z \in Z_\epsilon$ and $v \in V$. Note that isomorphic representations have the same central character, so assigning to a $\mathcal{U}_\epsilon$ module its central character give a well define map

$$\Xi : \mathrm{Rap}\,(\mathcal{U}_\epsilon) \to \mathrm{Spec}\,(Z_\epsilon)\,,$$

where $\mathrm{Rap}\,(\mathcal{U}_\epsilon)$ is the set of isomorphism classes of irreducible $\mathcal{U}_\epsilon$ modules, and $\mathrm{Spec}\,(Z_\epsilon)$ is the set of algebraic homomorphisms $Z_\epsilon \mapsto \mathbb{C}$.

To see that Ξ is surjective, let I^χ, for $\chi \in \mathrm{Spec}\,(Z_\epsilon)$, be the ideal in $\mathcal{U}_\epsilon$ generated by

$$\ker \chi = \{z - \chi(z) \cdot 1 : z \in Z_\epsilon\}\,.$$

To construct $V \in \Xi^{-1}(\chi)$ is the same as to construct an irreducible representation of the algebra $\mathcal{U}_\epsilon^\chi = \mathcal{U}_\epsilon / I^\chi$. Note that $\mathcal{U}_\epsilon^\chi$ is finite dimensional and non zero. Thus, we may take V, for example, has any irreducible subrepresentation of the regular representation of $\mathcal{U}_\epsilon^\chi$.

Composing with the surjective map $\mathrm{Spec}(Z_\epsilon) \to \mathrm{Spec}(Z_0)$, we obtain a surjective map

$$\Phi : \mathrm{Rap}(\mathcal{U}_\epsilon) \to \mathrm{Spec}(Z_0).$$

A priori in order to study representations one should study the representation theory of the algebra $\mathcal{U}_\epsilon^\chi$, for all $\chi \in \mathrm{Spec}(Z_0)$. However by [DCKP92] (or [DCP93] §16), we have that

Theorem 4.3.1. *Let χ_1 and $\chi_2 \in \mathrm{Spec}(Z_0)$ such that χ_1 and χ_2 live in the same symplectic leaf then $\mathcal{U}_\epsilon^{\chi_1} = \mathcal{U}_\epsilon^{\chi_2}$.*

4.4 Degree of $\mathcal{U}_\epsilon$.

Summarizing, if ϵ is a primitive l^{th} root of 1 with l odd and $l > d_i$ for all i, we have proved the following facts on $\mathcal{U}_\epsilon$:

- $\mathcal{U}_\epsilon$ is a domain (cf theorem 4.1.9),
- $\mathcal{U}_\epsilon$ is a finite module over Z_0(cf proposition 4.2.10).

Since the L.S. relations holds $\mathcal{U}_\epsilon$ (cf proposition 4.2.4), we can apply the theory developed in section 4.1.2, and we obtain that $\mathrm{Gr}\mathcal{U}_\epsilon$ is a twisted polynomial algebra, with some elements inverted. Hence all conditions of theorem 3.5.3 are verified, so

Theorem 4.4.1. *$\mathcal{U}_\epsilon$ is a maximal order.*

Proof. See [DCK90]. □

Note. As we have seen in section 3.5, since $\mathcal{U}_\epsilon$ is a maximal order, Z_ϵ is integrally closed and $\mathcal{U}_\epsilon \in \mathcal{C}_m$ for some $m \in \mathbb{N}$.

Following example 2.2.4, we can make the following construction: denote by $Q_\epsilon := Q(Z_\epsilon)$ the field of fraction of Z_ϵ, we have that $Q(\mathcal{U}_\epsilon) := \mathcal{U}_\epsilon \otimes_{Z_\epsilon} Q_\epsilon$ is a division algebra, finite dimensional over its center Q_ϵ. Denote by $\mathcal{F}$ the maximal commutative subfield of $Q(\mathcal{U}_\epsilon)$, we have that

(i) $\mathcal{F}$ is a finite extension of Q_ϵ of degree m,

(ii) $Q(\mathcal{U}_\epsilon)$ has dimension m^2 over Q_ϵ,

(iii) $Q(\mathcal{U}_\epsilon) \otimes_{Q_\epsilon} \mathcal{F} \cong M_m(\mathcal{F})$.

So,

Proposition 4.4.2. *There is a non empty closed proper subvariety $\mathcal{D}$ of* $\mathrm{Spec}(Z_\epsilon)$ *such that*

(i) If $\chi \in \mathrm{Spec}(Z_\epsilon) \setminus \mathcal{D}$, then U_ϵ^χ is isomorphic to $M_m(\mathbb{C})$, and (hence) there is, up to an isomorphism, exactly one irreducible U_ϵ module V_χ with character χ. One has $\dim V_\chi = m$.

(ii) If $\chi \in \mathcal{D}$, then $\dim \mathcal{U}_\epsilon^\chi \geq m^2$, but the dimension of every irreducible $\mathcal{U}_\epsilon^\chi$ module is strictly less than m.

Proof. Apply theorem 2.2.7. □

Note that, from the above discussion, we have:

$$\begin{aligned} \dim_{Q(Z_0)} Q_\epsilon &= \deg \tau \\ \dim_{Q_\epsilon} Q(\mathcal{U}_\epsilon) &= m^2 \\ \dim_{Q(Z_0)} Q(\mathcal{U}_\epsilon) &= l^{2N+n} \end{aligned}$$

where, the first equality is a definition, the second has been pointed out above and the third follows from the P.B.W theorem. Finally, we have

$$l^{2N+n} = m^2 \deg \tau.$$

Recall that, from proposition 4.2.12, we have $\deg \tau = l^n$, hence

$$\deg \mathcal{U}_\epsilon(\mathfrak{g}) = m = l^N.$$

4.5 Degree of $\mathcal{U}_\epsilon(\mathfrak{b})$

Recall that, we have $\mathcal{U}_q(\mathfrak{b}) = \mathcal{U}^{\geq 0} = \mathcal{U}^+\mathcal{U}^0 \subset \mathcal{U}_q(\mathfrak{g})$. We begin to give a more useful construction of $\mathcal{U}_q(\mathfrak{b})$. We have seen at the end of section 4.1.1 that for all $\omega \in \mathcal{W}$, we can construct two twisted derivation algebras $\mathcal{U}^\omega$ and $\mathcal{B}^\omega$.

Lemma 4.5.1. *Let $\omega_0 \in \mathcal{W}$ be the longest element, then $\mathcal{U}_q(\mathfrak{b}) = \mathcal{B}^{\omega_0}$.*

Proof. Follows from the definitions of $\mathcal{U}_q(\mathfrak{b})$ and $\mathcal{B}^{\omega_0}$. □

So, $\mathcal{U}_q(\mathfrak{b})$ is a twisted derivation algebra. Let ϵ be a primitive l^{th} root of 1 such that $l > d_i$ for all i, we may consider the specialized algebra

$$\mathcal{U}_\epsilon(\mathfrak{b}) = \mathcal{U}_q(\mathfrak{b})/(q-\epsilon) \subset \mathcal{U}_\epsilon(g).$$

Proposition 4.5.2. *(i) The monomials*

$$E_{\beta_1}^{k_1} \dots E_{\beta_N}^{k_n} K_1^{s_1} \dots K_n^{s_n}$$

for $(k_1, \dots, k_N) \in (\mathbb{Z}^+)^N$ and $(s_1, \dots, s_n) \in \mathbb{Z}^n$ form a $\mathbb{C}$ basis of $\mathcal{U}_\epsilon(\mathfrak{b})$

(ii) The L.S. relations holds in $\mathcal{U}_\epsilon(\mathfrak{b})$.

Proof. See [DCKP95] or [DCP93] §10. □

Using previous proposition and proposition 4.1.5, we have that $\mathcal{U}_\epsilon(\mathfrak{b})$ is a quasi derivation algebra with relations of type in example 3.2.8, so we can consider the associated quasi polynomial algebra $\overline{\mathcal{U}}_\epsilon(\mathfrak{b})$. We can then apply the theorem 3.2.9. We have

Theorem 4.5.3. *The algebras $\mathcal{U}_\epsilon(\mathfrak{b})$ and $\overline{\mathcal{U}}_\epsilon(\mathfrak{b})$ have the same degree.*

Note. We have that $\overline{\mathcal{U}}_\epsilon(\mathfrak{b}) \subset \mathrm{Gr}\,\mathcal{U}_\epsilon(\mathfrak{g})$.

So the algebra $\overline{\mathcal{U}}_\epsilon(\mathfrak{b})$ is a twisted polynomial algebra where the commutation relation are of type

$$x_i x_j = \epsilon^{h_{ij}} x_j x_i,$$

in order to compute is degree d is necessary to identify and study the corresponding matrix $H = (h_{ij})$ since, according to proposition 3.4.3, d^2 is equal to the number from elements of the image of H modulo l. Let us explicit the matrix H.

Let x_m denote the class of E_{β_m} for $m = 1, \dots, N$, then from relations 4.9, we have

$$x_i x_j = \epsilon^{(\beta_i|\beta_j)} x_j x_i, \text{ if } 0 < i < j$$

Thus we introduce the skew symmetric matrix $A = (a_{ij})$ with $a_{ij} = (\beta_i|\beta_j)$ if $i < j$.

Let k_i the class of K_i, relations 4.9 we obtain a $n \times N$ matrix

$$B = ((\omega_i|\beta_j))_{1 \leq i \leq n, 1 \leq j \leq N}$$

Let $t = 2$ unless the Cartan matrix is of type G_2 in which case $t = 6$, and let $\mathbb{Z}' = \mathbb{Z}\left[\frac{1}{t}\right]$. We wish to think the matrix A as the matrix of a skew form on a free $\mathbb{Z}'$ module V with basis $u_1, \ldots, u_N$. Identifying V with its dual V^* using the given basis, we may also think A as linear operator from V to itself.While we may think of the matrix B as a linear map from the module V to the module $Q^* \otimes_{\mathbb{Z}} \mathbb{Z}'$, where $Q^* = Hom_{\mathbb{Z}}(\Lambda, \mathbb{Z})$ be the dual lattice.

Construct the matrix T:

$$T = \begin{pmatrix} A & -^tB \\ B & 0 \end{pmatrix},$$

T is the matrix associated to the twisted polynomial algebra $\overline{\mathcal{U}}_\epsilon(\mathfrak{b})$. To study this the matrix we need the following

Lemma 4.5.4. *Let $w \in \mathcal{W}$ and fix a reduced expression $w = s_{i_1} \cdots s_{i_k}$. Given $\omega = \sum_{i=1}^n \delta_i \omega_i$, with $\delta_i = 0$ or 1. Set*

$$I_\omega(w) := \{t \in \{1, \ldots, k\} : s_{i_t}(\omega) \neq \omega\}.$$

Then

$$\omega - w(\omega) = \sum_{t \in I_\omega} \beta_t$$

Proof. We proceed by induction on the length of w. The hypothesis made implies $s_i(\omega) = \omega$ or $s_i(\omega) = \omega - \alpha_i$. Write $w = w' s_{i_k}$. If $k \notin I_\omega$, then $w(\omega) = w'(\omega)$ and we are done by induction. Otherwise

$$w(\omega) = w'(\omega - \alpha_{i_k}) = w'(\omega) - \beta_k$$

and again we are done by induction. □

Consider the operator $T_1 = \begin{pmatrix} A & -^tB \end{pmatrix}$ and $N = \begin{pmatrix} B & 0 \end{pmatrix}$ so that $T = T_1 \oplus N$.

Lemma 4.5.5. *(i) The operator T_1 is surjective*

(ii) The vector $v_\omega := \left(\sum_{t \in I_\omega} u_t\right) - \omega - w_0(\omega)$, as ω run thought the fundamental weights, form a basis of the kernel of T_1.

(iii) $N(v_\omega) = \omega - w_0(\omega) = \sum_{t \in I_\omega} \beta_t$.

Proof. See [DCKP95] or [DCP93] §10. □

Since T is the direct sum of T_1 and N, its kernel is the intersection of the two kernels of these operators. We have computed the kernel of T_1 in proposition 4.5.5, so the kernel of T is equal to the kernel of N restricted to the submodule spanned by vectors v_ω. Thus, we can identify N with the map $1 - w_0 : \Lambda \to Q$. At this point we need the following fact

Lemma 4.5.6. *Let $\theta = \sum_{i=1}^{n} a_i \alpha_i$ the highest root of the root system R. Let $\mathbb{Z}'' = \mathbb{Z}[a_1^{-1}, \dots, a_n^{-1}]$, and let $\Lambda'' = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}''$ and $Q'' = Q \otimes_{\mathbb{Z}} \mathbb{Z}''$. Then the $\mathbb{Z}''$ submodule $(1 - w_0)\Lambda''$ of Q'' is a direct summand.*

Proof. See [DCKP95] or [DCP93] §10. □

We call $l > 1$ a *good* integer if l is relative prime to t and to all the a_i

Theorem 4.5.7. *If l is a good integer, then*

$$\deg \mathcal{U}_\epsilon(\mathfrak{b}) = l^{\frac{1}{2}(l(\omega_0) + \mathrm{rank}(1-\omega_0))}$$

Proof. From the above discussion we see that $\deg \overline{\mathcal{U}}\_(\mathfrak{b}) = l^s$, where $s = (N + n) - (n - \mathrm{rank}(1 - w_0))$ with $N = l(w_0)$, which together with theorem 4.5.3 prove the claim. □

Note. This method for the calculation of the degree does not use the center of $\mathcal{U}_\epsilon(\mathfrak{b})$.

5. QUANTUM UNIVERSAL ENVELOPING ALGEBRAS FOR PARABOLIC LIE ALGEBRAS

Let ϵ be a primitive l^{th} root of the unity, we have seen in the previous chapter how the degree of $\mathcal{U}_\epsilon(\mathfrak{g})$ can be calculated. As we have seen in section 4.5, De Concini, Kaç and Procesi prove that $\mathcal{U}_\epsilon(\mathfrak{b})$ is a quasi polynomial algebra and they calculate the degree reducing itself to calculation of the rank of a matrix as we saw in chapter 4, theorem 4.5.7, they obtain the following formula

$$\deg \mathcal{U}_\epsilon(\mathfrak{b}) = l^{\frac{1}{2}(l(\omega_0)+\mathrm{rank}(1-\omega_0))}$$

In this chapter, we want to generalize the previous formula at $\mathfrak{p}$ parabolic subalgebra of $\mathfrak{g}$. If we wanted to follow the chosen direction for $\mathcal{U}_\epsilon(\mathfrak{g})$, we would have to determine the center $\mathcal{U}_\epsilon(\mathfrak{p})$. This seems to be a much greater and at the moment more open problem. Unfortunately $\mathcal{U}_\epsilon(\mathfrak{p})$ is not a quasi polynomial algebra, therefore we cannot directly apply the theory developed in chapter 3. So the main idea is to see $\mathcal{U}_\epsilon(\mathfrak{p})$ as a flat deformation of a suitable quasi polynomial algebra, and using it to calculate the degree.

We recall some notations. Let $\mathfrak{g}$ be a simple lie algebra, fix a Borel subgroup $\mathfrak{b} \subset \mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{g}$. Let $\mathfrak{p}$ be a parabolic subalgebra in standard position i.e. $\mathfrak{b} \subset \mathfrak{p}$. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ the Levi decomposition of $\mathfrak{p}$, with $\mathfrak{l}$ the Levi factor and $\mathfrak{u}$ the unipotent part.

Let R be the finite reduced root system associated to $\mathfrak{g}$, R^+ the positive roots and $\Pi \subset R^+$ the simple roots. Define $(R^\mathfrak{l})^+ = R^\mathfrak{l} \cap R^+$ and $\Pi^\mathfrak{l} = \Pi \cap R^+$ the positive respectively the simple roots of $\mathfrak{l}$ with respect to our choice. Set $C = (d_i a_{ij})$ and C' the Cartan matrix associated to $\mathfrak{g}$ and $\mathfrak{l}$, $\mathcal{W}$ the Weyl group associated to $\mathfrak{g}$ and $\mathcal{W}_\mathfrak{l} \subset \mathcal{W}$ the subgroup associated to $\mathfrak{l}$, Λ and $\Lambda^\mathfrak{l}$ the weight lattice and Q and $Q^\mathfrak{l}$ the root lattice of $\mathfrak{g}$ and $\mathfrak{l}$ respectively.

5.1 Parabolic quantum universal enveloping algebras

5.1.1 Definition of $\mathcal{U}(\mathfrak{p})$

Let $w_0^\mathfrak{l}$ be the longest element of $W_\mathfrak{l}$ and w_0 the longest element of W. We can choose a reduced expression $w_0 = s_{j_1} \dots s_{j_k} s_{i_1} \dots s_{i_h}$, such that $w_0^\mathfrak{l} = s_{i_1} \dots s_{i_h}$ is a reduced expression for $w_0^\mathfrak{l}$. We set $\overline{w} = w_0 (w_0^\mathfrak{l})^{-1} = s_{j_1} \dots s_{j_k}$, with $h = |(R^{\mathfrak{l}})^{+}|$ and $h + k = N = |R^{+}|$. We define, as in general case,

$$\begin{aligned} \beta_t^1 &= \overline{w} s_{i_1} \dots s_{i_{t-1}} (\alpha_{i_t}) \in (R^{\mathfrak{l}})^{+}, \\ \beta_t^2 &= s_{j_1} \dots s_{j_{t-1}} \left(\alpha_{i_{t+k}}\right) \in R^{+} \setminus (R^{\mathfrak{l}})^{+}. \end{aligned}$$

Let $\mathcal{U}$ the quantum group associated to $\mathfrak{g}$ define in 4.1.1, $\mathcal{B}$ the braid group associated to $\mathcal{W}$ with the Lusztig action over $\mathcal{U}$ define by 4.5. Given this choice, we obtain the q analogues of the root vectors:

$$\begin{aligned} E_{\beta_t^1} &= T_{\overline{w}} T_{i_1} \dots T_{i_{t-1}} (E_{i_t}), \\ E_{\beta_t^2} &= T_{j_1} \dots T_{j_{t-1}} \left(E_{i_{t+k}}\right). \end{aligned}$$

and

$$\begin{aligned} F_{\beta_t^1} &= T_{\overline{w}} T_{i_1} \dots T_{i_{t-1}} (F_{i_t}), \\ F_{\beta_t^2} &= T_{j_1} \dots T_{j_{t-1}} \left(F_{i_{t+k}}\right). \end{aligned}$$

The PBW theorem implies that, the monomials

$$E_{\beta_1^2}^{s_1} \cdots E_{\beta_k^2}^{s_k} E_{\beta_1^1}^{s_{k+1}} \cdots E_{\beta_h^1}^{s_{k+h}} K_\lambda F_{\beta_h^1}^{t_{k+h}} \cdots F_{\beta_1^1}^{t_{k+1}} F_{\beta_k^2}^{t_k} \cdots F_{\beta_1^2}^{t_1} \tag{5.1}$$

for $(s_1, \cdots, s_N)$, $(t_1, \dots, t_N) \in (\mathbb{Z}^{+})^N$ and $\lambda \in \Lambda$, form a $\mathbb{C}(q)$ basis of $\mathcal{U}$.

The choice of the reduced expression of w_0 and the LS relation for $\mathcal{U}$ implies that

Property. *For $i < j$ one has*

(i)

$$E_{\beta_j^1} E_{\beta_i^1} - q^{(\beta_i^1|\beta_j^1)} E_{\beta_i^1} E_{\beta_j^1} = \sum_{k \in \mathbb{Z}_{+}^N} c_k E_1^k$$

where $c_k \in \mathbb{C}(q)$ and $c_k \neq 0$ only when $k = (s_1, \dots, s_k)$ is such that $s_r = 0$ for $r \leq i$ and $r \geq j$, and $E_1^k = E_{\beta_1^1}^{s_1} \dots E_{\beta_k^1}^{s_k}$.

(ii)

$$E_{\beta_j^2} E_{\beta_i^2} - q^{-(\beta_i^2|\beta_j^2)} E_{\beta_i^2} E_{\beta_j^2} = \sum_{k \in \mathbb{Z}_{+}^N} c_k E_2^k$$

where $c_k \in \mathbb{C}(q)$ and $c_k \neq 0$ only when $k = (t_1, \dots, t_h)$ is such that $t_r = 0$ for $r \leq i$ and $r \geq j$, and $E_2^k = E_{\beta_1^2}^{t_1} \dots E_{\beta_h^2}^{t_h}$.

The same statement holds for $F_{\beta_i^1}$ and $F_{\beta_i^2}$.

The definition of the braid group action implies:

Property. *(i) If $i \in \Pi^{\mathfrak{l}}$, then*

$$\begin{aligned} E_i &= E_{\beta^1_s}, \\ F_i &= F_{\beta^1_s}. \end{aligned}$$

for some $s \in \{1, \ldots, h\}$.

(ii) If $i \in \Pi \setminus \Pi^{\mathfrak{l}}$, then

$$\begin{aligned} E_i &= E_{\beta^2_s}, \\ F_i &= F_{\beta^2_s}. \end{aligned}$$

for some $s \in \{1, \ldots, k\}$.

Definition 5.1.1. The simple connected quantum group associated to $\mathfrak{p}$, or *parabolic quantum group*, is the $\mathbb{C}(q)$ subalgebra of $\mathcal{U}(\mathfrak{g})$ generated by

$$\mathcal{U}(\mathfrak{p}) = \langle E_{\beta^1_i}, K_\lambda, F_{\beta_j} \rangle$$

for $i = 1, \ldots, h$, $j = 1 \ldots N$ and $\lambda \in \Lambda$.

Definition 5.1.2. 1. The quantum Levi factor of $\mathcal{U}(\mathfrak{p})$ is the subalgebra generated by

$$\mathcal{U}(\mathfrak{l}) = \langle E_{\beta^1_i}, K_\lambda, F_{\beta^1_i} \rangle$$

for $i = 1, \ldots, h$, and $\lambda \in \Lambda$.

2. The quantum unipotent part of $\mathcal{U}(\mathfrak{p})$ is the subalgebra generated by

$$\mathcal{U}^{\overline{w}} = \langle F_{\beta^2_s} \rangle$$

with $s = 1 \ldots h$

Set $\mathcal{U}^+(\mathfrak{p}) = \mathcal{U}^+(\mathfrak{l}) = \langle E_i \rangle_{i \in \Pi^{\mathfrak{l}}}$, $\mathcal{U}^-(\mathfrak{p}) = \langle F_i \rangle_{i \in \Pi}$, $\mathcal{U}^-(\mathfrak{l}) = \langle F_i \rangle_{i \in \Pi^{\mathfrak{l}}}$ and $\mathcal{U}^0(\mathfrak{p}) = \mathcal{U}^0(\mathfrak{l}) = \langle K_\lambda \rangle_{\lambda \in \Lambda}$. We have:

Property. *The definition of $\mathcal{U}(\mathfrak{p})$ and $\mathcal{U}(\mathfrak{l})$ is independent of the choice of the reduced expression of $w_0^{\mathfrak{l}}$ and w_0.*

Proof. Follows immediately from proposition 4.1.5. □

We can now state the P.B.W theorem for $\mathcal{U}(\mathfrak{p})$ and $\mathcal{U}(\mathfrak{l})$, which is an immediately consequence of 5.1.

Proposition 5.1.3. *(i) The monomials*

$$E^{s_1}_{\beta^1_1} \cdots E^{s_h}_{\beta^1_h} K_\lambda F^{t_{k+h}}_{\beta^1_h} \cdots F^{t_{k+1}}_{\beta^1_1} F^{t_k}_{\beta^2_k} \cdots F^{t_1}_{\beta^2_1}$$

for $(s_1, \cdots, s_h) \in (\mathbb{Z}^+)^h$, $(t_1, \ldots, t_N) \in (\mathbb{Z}^+)^N$ and $\lambda \in \Lambda$, form a $\mathbb{C}(q)$ basis of $\mathcal{U}(\mathfrak{p})$.

(ii) The monomials

$$E_{\beta_1^1}^{s_1}\cdots E_{\beta_h^1}^{s_h}K_\lambda F_{\beta_h^1}^{t_h}\cdots F_{\beta_1^1}^{t_1}$$

for $(s_1,\dots,s_h)$, $(t_1,\dots,t_h)\in(\mathbb{Z}^+)^h$ *and* $\lambda\in\Lambda$, *form a* $\mathbb{C}(q)$ *basis of* $\mathcal{U}(\mathfrak{l})$.

Proof. Follows easy from the P.B.W theorem for $\mathcal{U}_q(\mathfrak{g})$. □

Proposition 5.1.4. *Set* $m=\operatorname{rank}\mathfrak{l}=\#|\Pi^{\mathfrak{l}}|$. *The algebra* $\mathcal{U}(\mathfrak{p})$ *is generated by* E_i, F_j K_λ, *with* $i=1,\dots,m$, $j=1,\dots,n$ *and* $\lambda\in\Lambda$, *subject to the following relations:*

$$\begin{cases} K_\alpha K_\beta = K_{\alpha+\beta} \\ K_0 = 1 \end{cases} \tag{5.2}$$

$$\begin{cases} K_\alpha E_i K_{-\alpha} = q^{(\alpha|\alpha_i)}E_i \\ K_\alpha F_j K_{-\alpha} = q^{-(\alpha|\alpha_j)}F_j \end{cases} \tag{5.3}$$

$$[E_i,F_j]=\delta_{ij}\frac{K_{\alpha_i}-K_{-\alpha_i}}{q^{d_i}-q^{-d_i}} \tag{5.4}$$

$$\begin{cases} \sum\limits_{s=0}^{1-a_{ij}}(-1)^s\begin{bmatrix}1-a_{ij}\\ s\end{bmatrix}_{d_i}E_i^{1-a_{ij}-s}E_jE_i^s=0 \text{ if } i\neq j \\ \sum\limits_{s=0}^{1-a_{ij}}(-1)^s\begin{bmatrix}1-a_{ij}\\ s\end{bmatrix}_{d_i}F_i^{1-a_{ij}-s}F_jF_i^s=0 \text{ if } i\neq j. \end{cases} \tag{5.5}$$

Where $\begin{bmatrix}n\\ m\end{bmatrix}_{d_i}$ *is the* q *binomial coefficient defined in section 3.1.*

Proof. Follows from P.B.W. theorem and the L.S. relations. □

We state now some easy properties of $\mathcal{U}(\mathfrak{p})$:

Lemma 5.1.5. *The multiplication map*

$$\mathcal{U}^+(\mathfrak{l})\otimes\mathcal{U}^0(\mathfrak{l})\otimes\mathcal{U}^-(\mathfrak{l})\to\mathcal{U}(\mathfrak{l})$$

is an isomorphism of vector spaces.

Proof. Follows from the P.B.W theorem for $\mathcal{U}(\mathfrak{l})$. □

Lemma 5.1.6. *The multiplication map*

$$\mathcal{U}(\mathfrak{l})\otimes\mathcal{U}^{\overline{w}}\xrightarrow{m}\mathcal{U}(p)$$

defined by $m(x,u)=xu$ *for every* $x\in\mathcal{U}(\mathfrak{l})$ *and* $u\in\mathcal{U}^{\overline{w}}$, *is an isomorphism of vector spaces.*

Proof. The statement follows immediately from the proposition 5.1.3. □

Lemma 5.1.7. *The map $\mu : \mathcal{U}(\mathfrak{p}) \to \mathcal{U}(\mathfrak{l})$ defined by*

$$\mu\left(E_{\beta_1^1}^{s_1}\cdots E_{\beta_h^1}^{s_h}K_\lambda F_{\beta_h^1}^{t_{k+h}}\cdots F_{\beta_1^1}^{t_{k+1}}F_{\beta_k^2}^{t_k}\cdots F_{\beta_1^2}^{t_1}\right)$$
$$=\begin{cases} 0 & \text{if } t_i \neq 0 \text{ for some } i=1,\dots,h,\\ E_{\beta_1^1}^{s_1}\cdots E_{\beta_h^1}^{s_h}K_\lambda F_{\beta_h^1}^{t_{k+h}}\cdots F_{\beta_1^2}^{t_{k+1}} & \text{if } t_i = 0 \text{ for all } i=1,\dots,h,\end{cases}$$

is an homomorphism of algebras.

Proof. Simple verification of the definition. □

Proposition 5.1.8. *$\mathcal{U}(\mathfrak{p})$ and $\mathcal{U}(\mathfrak{l})$ are Hopf subalgebra of $\mathcal{U}$.*

Proof. Follows immediately from the definition. □

5.1.2 Definition of $\mathcal{U}_\epsilon(\mathfrak{p})$

Let $\mathcal{A} = \mathbb{C}[q, q^{-1}]$, and $\mathcal{U}_{\mathcal{A}}$ the integral form of $\mathcal{U}$ defined by 4.2.2. Like in the general case, we define $\mathcal{U}_{\mathcal{A}}(\mathfrak{p})$, has the subalgebra generated by $E_{\beta_i^1}$, $F_{\beta_i^1}$, $F_{\beta_s^2}$, $K_j^{\pm 1}$ and L_j, with $i = 1, \dots, h$, $s = 1, \dots, k$ and $j = 1, \dots, n$.

Definition 5.1.9. Let $\epsilon \in \mathbb{C}$, we define

$$\mathcal{U}_\epsilon(\mathfrak{p}) = \mathcal{U}_{\mathcal{A}}(\mathfrak{p}) \otimes_{\mathcal{A}} \mathbb{C}$$

using the homomorphism $\mathcal{A} \to \mathbb{C}$ tacking $q \to \epsilon$

Let $\epsilon \in \mathbb{C}$ such that $\epsilon^{2d_i} \neq 1$ for all i, then

Property. *$\mathcal{U}_\epsilon(\mathfrak{p}) \subset \mathcal{U}_\epsilon(\mathfrak{g})$. Moreover $\mathcal{U}_\epsilon(\mathfrak{p})$ is generated by $E_{\beta_i^1}$, F_{β_s} and $K_j^{\pm 1}$, for $i = 1, \dots, h$, $s = 1, \dots, N$ and $j = 1, \dots, n$.*

Proof. The claim is a consequence of the definition of $\mathcal{U}_{\mathcal{A}}(\mathfrak{p})$. □

Proposition 5.1.10. *The P.B.W. theorem and the L.S. relations holds for $\mathcal{U}_\epsilon(\mathfrak{p})$*

Proof. The claim is a consequence of the P.B.W. theorem and L.S. relation for $\mathcal{U}_\epsilon(\mathfrak{g})$ and the choice of the decomposition of the reduced expression of w_0. □

5.2 The center of $\mathcal{U}_\epsilon(\mathfrak{p})$

Let $\epsilon \in \mathbb{C}$ a primitive l^{th} root of unity, with l odd and $l > d_i$ for all i. Has in section 4.2.1, we note that at root of unity the algebra $\mathcal{U}_\epsilon(\mathfrak{p})$ has a big center. The aim of this section is to extend some properties of the center of $\mathcal{U}_\epsilon$ at the center of $\mathcal{U}_\epsilon(\mathfrak{p})$.

Proposition 5.2.1. *For $i = 1, \ldots, k$, $s = 1, \ldots, h$ and $j = 1, \ldots, n$, $E^l_{\beta^1_i}$, $F^l_{\beta^1_i}$, $F^l_{\beta^2_s}$ and $K^{\pm l}_j$ lie in the center of $\mathcal{U}_\epsilon(\mathfrak{p})$*

Proof. Its well known that these elements lie in the center of $\mathcal{U}_\epsilon$ (cf [Lus93] of [DCP93]), but they also lie in $\mathcal{U}_\epsilon(\mathfrak{p})$, hence the claim. □

For $\alpha \in (R^{\mathfrak{l}})^+$, $\beta \in R^+$ and $\lambda \in Q$, define $e_\alpha = E^l_\alpha$, $f_\beta = F^l_\beta$, $k^{\pm 1}_\lambda = K^{\pm l}_\lambda$. Let $Z_0(\mathfrak{p})$ be the subalgebra generated by the e_α, f_β and $k^{\pm 1}_i$.

Proposition 5.2.2. *Let Z^0_0, Z^+_0 and Z^-_0 be the subalgebra generated by $k^{\pm 1}_i$, e_α and f_β respectively.*

(i) $Z^\pm_0 \subset \mathcal{U}^\pm_\epsilon(\mathfrak{p})$

(ii) Multiplication defines an isomorphism of algebras

$$Z^-_0 \otimes Z^0_0 \otimes Z^+_0 \to Z_0(\mathfrak{p})$$

(iii) Z^0_0 is the algebra of Laurent polynomial in the k_i, and Z^+_0 and Z^-_0 are polynomial algebra with generators e_α and f_β respectively.

(iv) $\mathcal{U}_\epsilon(\mathfrak{p})$ is a free $Z^0_\epsilon(\mathfrak{p})$ module with basis the set of monomial

$$E^{s_1}_{\beta^1_1} \cdots E^{s_h}_{\beta^1_h} K^{r_1}_1 \cdots K^{r_n}_n F^{t_{k+h}}_{\beta^1_h} \cdots F^{t_{k+1}}_{\beta^1_1} F^{t_k}_{\beta^2_k} \cdots F^{t_1}_{\beta^2_1}$$

for which $0 \le s_j, t_i, r_v < l$

Proof. By definition of $\mathcal{U}^+(\mathfrak{p})$, we have $e_\alpha \in \mathcal{U}^+(\mathfrak{p})$, since $\mathcal{U}^+(\mathfrak{p})$ is a subalgebra (i) follows. (ii) and (iii) are easy corollary of the definitions and of the P.B.W. theorem. (iv) follows from the P.B.W. theorem for $\mathcal{U}(\mathfrak{p})$. □

The preceding proposition shows that $\mathcal{U}_\epsilon(\mathfrak{p})$ is a finite $Z_0(\mathfrak{p})$ module. Since Z_0 is clearly noetherian, from (iii), it follows that $Z_\epsilon(\mathfrak{p}) \subset \mathcal{U}_\epsilon(\mathfrak{p})$ is a finite $Z_0(\mathfrak{p})$ module, and hence integral over $Z_0(\mathfrak{p})$. By the Hilbert basis theorem $Z_\epsilon(\mathfrak{p})$ is a finitely generated algebra. Thus the affine schemes $\mathrm{Spec}(Z_\epsilon(\mathfrak{p}))$ and $\mathrm{Spec}(Z_0(\mathfrak{p}))$ are algebraic varieties. Note that $\mathrm{Spec}(Z_0)$ is isomorphic to $\mathbb{C}^N \times \mathbb{C}^{l(h)} \times (\mathbb{C}^*)^n$. Moreover the inclusion $Z_0(\mathfrak{p}) \hookrightarrow Z_\epsilon(\mathfrak{p})$ induces a projection $\tau : \mathrm{Spec}(Z_\epsilon(\mathfrak{p})) \to \mathrm{Spec}(Z_0(\mathfrak{p}))$, and we have

Proposition 5.2.3. $\mathrm{Spec}(Z_\epsilon(\mathfrak{p}))$ *is an affine variety and τ is a finite surjective map.*

Proof. Follows from the Cohen-Seidenberg theorem ([Ser65] ch. III). □

We conclude this section by discussing the relation between the center and the Hopf algebra structure of $\mathcal{U}_\epsilon(\mathfrak{p})$.

Proposition 5.2.4. *(i) $Z_0(\mathfrak{p})$ is a Hopf subalgebra of $\mathcal{U}_\epsilon(\mathfrak{p})$.*

(ii) $Z_0(\mathfrak{p})$ *is a Hopf subalgebra of* Z_0.

(iii) $Z_0(\mathfrak{p}) = Z_0 \cap \mathcal{U}_\epsilon(\mathfrak{p})$.

Proof. It follows directly from the given definitions. □

The fact that $Z_0(\mathfrak{p})$ is an Hopf algebra tells us that $\mathrm{Spec}(Z_0(\mathfrak{p}))$ is an algebraic group . Moreover, the inclusion $Z_0(\mathfrak{p}) \hookrightarrow Z_0$ being an inclusion of Hopf algebras, induces a group homomorphism,

$$\mathrm{Spec}(Z_0) \to \mathrm{Spec}(Z_0(\mathfrak{p})).$$

Let us recall that in theorem (i), we have that

$$\mathrm{Spec}(Z_0) = \left\{(a,b) :\in B^- \times B^+ : \pi^-(a)\pi^+(b) = 1\right\}$$

where $\pi^\pm : B^\pm \to H$. From this and, the explicit description of the subalgebra $Z_0(\mathfrak{p}) \subset Z_0$, we get

$$\mathrm{Spec}(Z_0(\mathfrak{p})) = \left\{(a,b) :\in B_L^- \times B^+ : \pi^-(a)\pi^+(b) = 1\right\}$$

where $L \subset G$ is the connected subgroup of G such that $LieL = \mathfrak{l}$, and $B_L^- = B^- \cap L$.

5.3 Parametrization of irreducible representations

5.3.1 Character of a representation

We begin by observing that every irreducible $\mathcal{U}_\epsilon(\mathfrak{p})$ module V is finite dimensional. Indeed, let $\mathcal{Z}(V)$ be the subalgebra of the algebra of intertwining operators of V generated by the action of the elements in $Z_\epsilon(\mathfrak{p})$. Since $\mathcal{U}_\epsilon(\mathfrak{p})$ is finitely generated as a $Z_\epsilon(\mathfrak{p})$ module, V is finitely generated as $\mathcal{Z}(V)$ module. If $0 \neq f \in \mathcal{Z}(V)$, then $f \cdot V = V$, otherwise $f \cdot V$ is a proper submodule V. Hence, by Nakayama's lemma, there exist an endomorphism $g \in \mathcal{Z}(V)$ such that $1 - gf = 0$, i.e. f is invertible. Thus $\mathcal{Z}(V)$ is a field. It follows easily that $\mathcal{Z}(V)$ consists of scalar operators. Thus V is a finite dimensional vector space.

Since $Z_\epsilon(\mathfrak{p})$ acts by scalar operators on V, there exists an homomorphism $\chi_V : Z_\epsilon \mapsto \mathbb{C}$, the *central character of* V, such that

$$z \cdot v = \chi_V(z)v$$

for all $z \in Z_\epsilon$ and $v \in V$. Note that isomorphic representations have the same central character, so assigning to a $\mathcal{U}_\epsilon(\mathfrak{p})$ module its central character gives a well define map

$$\Xi : \mathrm{Rap}\left(\mathcal{U}_\epsilon(\mathfrak{p})\right) \to \mathrm{Spec}\left(Z_\epsilon(\mathfrak{p})\right),$$

where Rap $(\mathcal{U}_\epsilon(\mathfrak{p}))$ is the set of isomorphism classes of irreducible $\mathcal{U}_\epsilon(\mathfrak{p})$ modules, and Spec $(Z_\epsilon(\mathfrak{p}))$ is the set of algebraic homomorphisms $Z_\epsilon(\mathfrak{p}) \mapsto \mathbb{C}$.

To see that Ξ is surjective, let I^χ, for $\chi \in \mathrm{Spec}\,(Z_\epsilon(\mathfrak{p}))$, be the ideal in $\mathcal{U}_\epsilon(\mathfrak{p})$ generated by

$$\ker \chi = \{z - \chi(z) \cdot 1 : z \in Z_\epsilon(\mathfrak{p})\} .$$

To construct $V \in \Xi^{-1}(\chi)$ is the same as to construct an irreducible representation of the algebra $\mathcal{U}_\epsilon^\chi(\mathfrak{p}) = \mathcal{U}_\epsilon(\mathfrak{p})/I^\chi$. Note that $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ is finite dimensional and non zero. Thus, we may take V, for example, to be any irreducible subrepresentation of the regular representation of $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$.

Let $\chi \in \mathrm{Spec}(Z_0(\mathfrak{p}))$, we define,

$$\mathcal{U}_\epsilon^\chi(\mathfrak{p}) = \mathcal{U}_\epsilon(\mathfrak{p})/J^\chi$$

where J^χ is the two sided ideal generated by

$$\ker \chi = \{z - \chi(z) \cdot 1 : z \in Z_0(\mathfrak{p})\}$$

5.4 A deformation to a quasi polynomial algebra

In this section we construct the main tool of this thesis. We want to modify the relation that define the non restricted integral form of $\mathcal{U}_\epsilon(\mathfrak{g})$, so as to obtain a deformation of $\mathcal{U}_\epsilon(\mathfrak{p})$ to a quasi-polynomial algebra. We begin by constructing the deformation for $\mathcal{U}_\epsilon(\mathfrak{g})$, which is exactly a reformulation of the construction of Gr$\,\mathcal{U}$ in section 4.1.2.

5.4.1 The case $\mathfrak{p} = \mathfrak{g}$.

Definition 5.4.1. Let $t \in \mathbb{C}$, we define $\mathcal{U}_\epsilon^t$ the algebra over $\mathbb{C}$ on generators E_i, F_i, L_i and $K_i^{\pm}$, for $i = 1, \ldots, n$, subject to the following relation:

$$\begin{cases} K_i^{\pm 1} K_j^{\pm 1} = K_j^{\pm 1} K_i^{\pm 1} \\ K_i K_i^{-1} = 1 \end{cases} \tag{5.6}$$

$$\begin{cases} K_i\,(E_j)\,K_i^{-1} = \epsilon^{a_{ij}} E_j \\ K_i\,(F_j)\,K_i^{-1} = \epsilon^{-a_{ij}} F_j \end{cases} \tag{5.7}$$

$$\begin{cases} [E_i, F_i] = t\delta_{ij} L_i \\ \left(ad_{\sigma_{-\alpha_i}} E_i\right)^{1-a_{ij}} E_j = 0 \\ \left(ad_{\sigma_{-\alpha_i}} F_i\right)^{1-a_{ij}} F_j = 0 \end{cases} \tag{5.8}$$

$$\begin{cases} \left(\epsilon^{d_i} - \epsilon^{-d_i}\right) L_i = t\left(K_i - K_i^{-1}\right) \\ [L_i, E_j] = t\frac{\epsilon^{a_{ij}}-1}{\epsilon^{d_i}-\epsilon^{-di}}\left(E_j K_i + K_i^{-1} E_j\right) \\ [L_i, F_j] = t\frac{\epsilon^{-a_{ij}}-1}{\epsilon^{d_i}-\epsilon^{-di}}\left(F_j K_i + K_i^{-1} F_j\right) \end{cases} \tag{5.9}$$

Proposition 5.4.2. *For $t = 1$, we have $\mathcal{U}_\epsilon^1 = \mathcal{U}_\epsilon$*

Proof. For $t = 1$, the relations of $\mathcal{U}_\epsilon^1$ are exactly the relation 4.2.2 that define $\mathcal{U}_\epsilon$. □

Let $0 \neq \lambda \in \mathbb{C}$, define

$$\vartheta_\lambda(E_i) = \frac{1}{\lambda}E_i,\ \vartheta_\lambda(F_i) = \frac{1}{\lambda}F_i,\ \vartheta_\lambda(L_i) = \frac{1}{\lambda}L_i,\ \vartheta_\lambda(K_i^{\pm 1}) = K_i^{\pm 1}, \tag{5.10}$$

for $i = 1, \ldots, n$.

Proposition 5.4.3. *For any $0 \neq \lambda \in \mathbb{C}$, ϑ_λ is an isomorphism of algebra between $\mathcal{U}_\epsilon^t$ and $\mathcal{U}_\epsilon^{\lambda t}$*

Proof. Simple verification of the property. □

Set $\mathcal{S}_\epsilon := \mathcal{U}_\epsilon^{t=0}$, we want to construct an explicit realization of it. Let $\mathcal{D} = \mathcal{U}_\epsilon(\mathfrak{b}_+) \otimes \mathcal{U}_\epsilon(\mathfrak{b}_-)$ and define the map

$$\Sigma : \mathcal{S}_\epsilon \to \mathcal{D}$$

by $\Sigma(E_i) = \mathcal{E}_i := E_i \otimes 1$, $\Sigma(F_i) = \mathcal{F}_i = 1 \otimes F_i$, and $\Sigma(K_i^{\pm 1}) = \mathcal{K}_i^{\pm 1} := K_i^{\pm 1} \otimes K_i^{\pm 1}$ for $i = 1, \ldots, n$.

Lemma 5.4.4. *Σ is a well defined map.*

Proof. We must verify that the image of E_i, F_i and K_i satisfy the relation 5.6 for $t = 0$.

$$\begin{cases} \mathcal{K}_i^{\pm}\mathcal{K}_j^{\pm} = \mathcal{K}_j^{\pm}\mathcal{K}_i^{\pm} \\ \mathcal{K}_i\mathcal{K}_i^{-1} = 1 \end{cases} \tag{5.11}$$

$$\begin{cases} \mathcal{K}_i\mathcal{E}_j\mathcal{K}_i^{-1} = \epsilon^{a_{ij}}\mathcal{E}_j \\ \mathcal{K}_i\mathcal{F}_j\mathcal{K}_i^{-1} = \epsilon^{-a_{ij}}\mathcal{F}_j \end{cases} \tag{5.12}$$

$$\begin{cases} [\mathcal{E}_i, \mathcal{F}_i] = 0 \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{\epsilon_i} \mathcal{E}_i^{1-a_{ij}-r}\mathcal{E}_j\mathcal{E}_i^r = 0 \\ \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{\epsilon_i} \mathcal{F}_i^{1-a_{ij}-r}\mathcal{F}_j\mathcal{F}_i^r = 0 \end{cases} \tag{5.13}$$

Note that the relations 5.11 are obvious. We begin by demonstrating the relation 5.12

$$\begin{aligned} \mathcal{K}_i\mathcal{E}_j\mathcal{K}_i^{-1} &= K_i \otimes K_i\,(E_i \otimes 1)\,K_i^{-1} \otimes K_i^{-1} \\ &= K_iE_iK_i^{-1} \otimes 1 \\ &= \epsilon^{a_{ij}}E_i \otimes 1 \\ &= \epsilon^{a_{ij}}\mathcal{E}_i \end{aligned}$$

in the same way, we have $\mathcal{K}_i\mathcal{F}_j\mathcal{K}_i^{-1} = \epsilon^{-a_{ij}}\mathcal{F}_i$. Finally we have

$$\begin{aligned}
[\mathcal{E}_i, \mathcal{F}_j] &= \mathcal{E}_i\mathcal{F}_j - \mathcal{F}_j\mathcal{E}_i \\
&= (E_i \otimes 1)(1 \otimes F_j) - (1 \otimes F_j)(E_i \otimes 1) \\
&= E_i \otimes F_j - E_i \otimes F_j \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{\epsilon_i} & \mathcal{E}_i^{1-a_{ij}-r}\mathcal{E}_j\mathcal{E}_i^r \\
&= \sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{e_i} E_i^{1-a_{ij}-r}E_jE_i^r \otimes 1 \\
&= 0
\end{aligned}$$

in the same way, we can verify the third relation of 5.13. □

Note that Σ is injective, then we can identify $\mathcal{S}_\epsilon$ with the subalgebra of $\mathcal{D}$ generated by $\mathcal{E}_i$, $\mathcal{F}_i$ and $\mathcal{K}_i$, for $i = 1, \ldots, n$. We define now the analogues of the root vectors for $\mathcal{S}_\epsilon$:

Definition 5.4.5. For all $i = 1, \ldots, N$, let

(i) $\mathcal{E}_{\beta_i} := E_{\beta_i} \otimes 1 \in \mathcal{S}_\epsilon$

(ii) $\mathcal{F}_{\beta_i} := 1 \otimes F_{\beta_i} \in \mathcal{S}_\epsilon$

As a consequence of this we get a P.B.W theorem for $\mathcal{S}_\epsilon$.

Proposition 5.4.6. *The monomials*

$$\mathcal{E}_{\beta_1}^{k_1} \ldots \mathcal{E}_{\beta_N}^{k_N}\mathcal{K}_1^{s_1} \ldots \mathcal{K}_n^{s_n}\mathcal{F}_{\beta_N}^{h_1} \ldots \mathcal{F}_{\beta_1}^{k_1}$$

for $(k_1, \ldots, k_N)$, $(h_1, \ldots, h_N) \in (\mathbb{Z}^+)^N$ *and* $(s_1, \ldots, s_n) \in \mathbb{Z}^n$, *form a* $\mathbb{C}$ *basis of* $\mathcal{S}_\epsilon$. *Moreover*

$$\mathcal{S}_\epsilon = \mathcal{S}_\epsilon^- \otimes \mathcal{S}_\epsilon^0 \otimes \mathcal{S}_\epsilon^+$$

where $\mathcal{S}_\epsilon^+$ *(resp.* $\mathcal{S}_\epsilon^-$ *and* $\mathcal{S}_\epsilon^0$*) is the subalgebra generated by* $\mathcal{E}_{\beta_i}$ *(resp.* $\mathcal{F}_{\beta_i}$ *and* $\mathcal{K}_i$*).*

Proof. This follows from the injectivity of Σ and proposition 4.5.2 □

Note. Its is clear that $\mathcal{E}_{\beta_i}$ is also the image of the element $E_{\beta_i} \in \mathcal{U}_\epsilon^t$, where the E_{β_i} are non commutative polynomials in the E_i's by Lusztig procedure ([Lus93]). The same thing is true for $\mathcal{F}_{\beta_i}$ and F_{β_i}.

We see now, that the L.S. relation holds for $\mathcal{S}_\epsilon$.

Proposition 5.4.7. *For $i < j$ one has*

1.
$$\mathcal{E}_{\beta_j}\mathcal{E}_{\beta_i} - \epsilon^{(\beta_i|\beta_j)}\mathcal{E}_{\beta_i}\mathcal{E}_{\beta_j} = \sum_{k\in\mathbb{Z}_+^N} c_k\mathcal{E}^k \tag{5.14}$$

where $c_k \in \mathbb{C}$ and $c_k \neq 0$ only when $k = (k_1, \dots, k_N)$ is such that $k_s = 0$ for $s \leq i$ and $s \geq j$, and $\mathcal{E}^k = \mathcal{E}_{\beta_1}^{k_1} \dots \mathcal{E}_{\beta_N}^{k_N}$.

2.
$$\mathcal{F}_{\beta_j}\mathcal{F}_{\beta_i} - \epsilon^{-(\beta_i|\beta_j)}\mathcal{F}_{\beta_i}\mathcal{F}_{\beta_j} = \sum_{k\in\mathbb{Z}_+^N} c_k\mathcal{F}^k \tag{5.15}$$

where $c_k \in \mathbb{C}$ and $c_k \neq 0$ only when $k = (k_1, \dots, k_N)$ is such that $k_s = 0$ for $s \leq i$ and $s \geq j$, and $\mathcal{F}^k = \mathcal{F}_{\beta_N}^{k_N} \dots \mathcal{F}_{\beta_1}^{k_1}$.

Proof. We have by definition $\mathcal{E}_{\beta_i} = E_{\beta_i} \otimes 1$, then

$$\begin{aligned}\mathcal{E}_{\beta_j}\mathcal{E}_{\beta_i} - \epsilon^{(\beta_i|\beta_j)}\mathcal{E}_{\beta_i}\mathcal{E}_{\beta_j} &= \left(E_{\beta_j}E_{\beta_i} - \epsilon^{(\beta_i|\beta_j)}E_{\beta_i}E_{\beta_j}\right) \otimes 1 \\ &= \left(\sum_{k\in\mathbb{Z}_+^N} c_k E^k\right) \otimes 1\end{aligned}$$

where we have been using the L.S. relation for the E_{β_i}. Note now that

$$\mathcal{E}^k = E^k \otimes 1,$$

then

$$\begin{aligned}\mathcal{E}_{\beta_j}\mathcal{E}_{\beta_i} - \epsilon^{(\beta_i|\beta_j)}\mathcal{E}_{\beta_i}\mathcal{E}_{\beta_j} &= \left(\sum_{k\in\mathbb{Z}_+^N} c_k E^k\right) \otimes 1 \\ &= \sum_{k\in\mathbb{Z}_+^N} c_k\mathcal{E}^k\end{aligned}$$

In the same way, we can prove the L.S. relation for the $\mathcal{F}_{\beta_i}$ □

Theorem 5.4.8. *(i) $\mathcal{S}_\epsilon = \mathcal{U}_\epsilon^{t=0}$ is a twisted derivation algebra.*

(ii) $\mathrm{Gr}\,\mathcal{U}_\epsilon$, where $\mathrm{Gr}\,\mathcal{U}_\epsilon$ is defined by the relation 4.9, is a degeneration, in the sense of twisted derivation algebra, of $\mathcal{S}_\epsilon$

Proof. Define $\mathcal{U}^0 = \mathbb{C}[\mathcal{E}_{\beta_1}, \mathcal{F}_{\beta_N}] \subset \mathcal{S}_\epsilon$, then we can define

$$\mathcal{U}^i = \mathcal{U}_{\sigma,D}^{i-1}\left[\mathcal{E}_{\beta_i}, \mathcal{F}_{\beta_{N-i}}\right] \subset \mathcal{S}_\epsilon$$

where σ and D are given by the L.S. relation. Note now that, the $\mathcal{K}_i$, for $i = 1, \dots, n$ normalize $\mathcal{U}^N$, and when we add them to this algebra we perform an iterated construction of twisted Laurent polynomial. The resulting algebra will be called $\mathcal{T}$. We now claim

$$\mathcal{S}_\epsilon = \mathcal{T}$$

Note that, by construction $\mathcal{T} \subset \mathcal{S}_\epsilon$, so we only have to prove that $\mathcal{S}_\epsilon \subset \mathcal{T}$. Now note that

$$\mathcal{E}_{\beta_1}^{k_1} \cdots \mathcal{E}_{\beta_N}^{k_N} \mathcal{K}_1^{s_1} \cdots \mathcal{K}_n^{s_n} \mathcal{F}_{\beta_N}^{h_N} \cdots \mathcal{F}_{\beta_1}^{h_1} \in \mathcal{T}$$

for every $(k_1, \ldots, k_N)$, $(h_1, \ldots, h_N) \in (\mathbb{Z}^+)^N$ and $(s_1, \ldots, s_n) \in \mathbb{Z}^n$. Then by proposition 5.4.6 we have $\mathcal{S}_\epsilon \subset \mathcal{T}$.

The second part is obtained by using standard technique of quasi polynomial algebra. Denote by $\overline{\mathcal{S}}_\epsilon$ the quasi polynomial algebra associated to $\mathcal{S}_\epsilon$. It easy to see that $\overline{\mathcal{S}}_\epsilon \cong \mathrm{Gr}\,\mathcal{U}_\epsilon$. □

We finish this section with some remarks on the center of $\mathcal{U}_\epsilon^t$. Recall that $\mathcal{U}_\epsilon^t$ is isomorphic to $\mathcal{U}_\epsilon$ for every $t \in \mathbb{C}^*$, hence Z_ϵ^t is isomorphic to $Z_\epsilon^1 = Z_\epsilon$. For $t = 0$, we define C_0 the subalgebra of $\mathcal{S}_\epsilon$ generated by $\mathcal{E}_\beta^l$, $\mathcal{F}_\beta^l$ for $\beta \in R^+$ and $\mathcal{K}_j^{\pm l}$ for $j = 1, \ldots, n$ and let C_ϵ be the center of $\mathcal{S}_\epsilon$. Let $Z_0[t]$ the trivial deformation of Z_0

Lemma 5.4.9. *(i) $\rho : Z_0[t] \to \mathcal{U}_\epsilon^t$ defined in the obvious way is an injective homomorphism of algebra.*

(ii) $\mathcal{U}_\epsilon^t$ is a free $Z_0[t]$ module with base the set of monomials

$$E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N} K_1^{s_1} \cdots K_n^{s_n} F_{\beta_N}^{h_N} \cdots F_{\beta_1}^{h_1}$$

for which $0 \le k_i, s_j, h_i < l$, for $i = 1, \ldots, N$ and $j = 1, \ldots, n$.

Proof. (i) follows by definitions of $Z_0[t]$. (ii) follows from the P.B.W theorem. □

Lemma 5.4.10. *$Z_0[t]/(t) \cong C_0$ and $Z_0[t]/(t-1) \cong Z_0$.*

Proof. Follows from the definitions of $Z_0[t]$, C_0 and Z_0 □

Proposition 5.4.11. *$\mathcal{U}_\epsilon$ and $\mathcal{S}_\epsilon$ are isomorphic has Z_0 modules.*

Proof. Follows from the previous lemma. □

5.4.2 general case

We can now study the general case.

Definition 5.4.12. Let $\mathcal{U}_\epsilon^t(\mathfrak{p})$ be the subalgebra of $\mathcal{U}_\epsilon^t$ generated by $E_{\beta_i^1}$, F_{β_j} and $K_s^{\pm 1}$ for $i = 1, \ldots, h$, $j = 1, \ldots, N$ and $s = 1, \ldots, n$.

Set $\mathcal{S}_\epsilon(\mathfrak{p}) = \mathcal{U}_\epsilon^{t=0}(\mathfrak{p}) \subset \mathcal{S}_\epsilon$.

Proposition 5.4.13. *(i) For every $t \in \mathbb{C}$, $\mathcal{U}_\epsilon^t(\mathfrak{p})$ is an Hopf subalgebra of $\mathcal{U}_\epsilon^t$.*

(ii) For any $\lambda \neq 0$, ϑ_λ defines by 5.10 is an isomorphism of algebra between $\mathcal{U}_\epsilon^t(\mathfrak{p})$ and $\mathcal{U}_\epsilon^{\lambda t}(\mathfrak{p})$.

Proof. This is an immediate consequence of the same property in the case $\mathfrak{p} = \mathfrak{g}$. □

We can now state the main theorem of this section

Theorem 5.4.14. *$\mathcal{S}_\epsilon(\mathfrak{p})$ is a twisted derivation algebra*

Proof. We use the same technique as we used on the proof of theorem 5.4.8. Using the notation of section 4.5, let $\mathcal{D}(\mathfrak{p}) = \mathcal{U}_\epsilon(\mathfrak{b}_+^{\mathfrak{l}}) \otimes \mathcal{U}_\epsilon(\mathfrak{b}_-)$. Define

$$\Sigma : \mathcal{S}_\epsilon(\mathfrak{p}) \to \mathcal{D}(\mathfrak{p})$$

by $\Sigma(E_i) = \mathcal{E}_i$, $\Sigma(F_j) = \mathcal{F}_j$, $\Sigma(K_j^{\pm 1}) = \mathcal{K}_j^{\pm 1}$ for $i \in \Pi^e l$ and $j = 1, \ldots, n$.

Lemma 5.4.15. *$\mathcal{S}_\epsilon(\mathfrak{p})$ is a subalgebra of $\mathcal{S}_\epsilon$*

Proof. Note that $\mathcal{D}(\mathfrak{p})$ is a subalgebra of $\mathcal{D}$, and, as in lemma 5.4.4, the map Σ is well define and injective. So, we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{S}_\epsilon(\mathfrak{p}) & \xrightarrow{\Sigma} & \mathcal{D}(\mathfrak{p}) \\ \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle j} \\ \mathcal{S}_\epsilon & \xrightarrow{\Sigma} & \mathcal{D} \end{array}$$

Since Σ and j are injective map, we have that i is also injective □

So we can identify $\mathcal{S}_\epsilon(\mathfrak{p})$ with the subalgebra of $\mathcal{S}_\epsilon$ generated by $\mathcal{E}_{\beta_i^1}$, $\mathcal{F}_{\beta_s}$ and $\mathcal{K}_j^{\pm 1}$ for $i = 1, \ldots, h$, $s = 1, \ldots, N$ and $j = 1, \ldots, n$. As corollary of proposition 5.4.6 and proposition 5.4.7, we have:

Proposition 5.4.16. *(i) The monomials*

$$\mathcal{E}_{\beta_1^1}^{k_1} \ldots \mathcal{E}_{\beta_h^1}^{k_h} \mathcal{K}_1^{s_1} \ldots \mathcal{K}_n^{s_n} \mathcal{F}_{\beta_N}^{t_1} \ldots \mathcal{F}_{\beta_1}^{t_1}$$

for $(k_1, \ldots, k_h) \in (\mathbb{Z}^+)^h$, $(t_1, \ldots, t_N) \in (\mathbb{Z}^+)^N$ and $(s_1, \ldots, s_n) \in \mathbb{Z}^n$, form a $\mathbb{C}$ basis of $\mathcal{S}_\epsilon$.

(ii) For $i < j$ one has

(a)

$$\mathcal{E}_{\beta_j^1}\mathcal{E}_{\beta_i^1} - \epsilon^{(\beta_i^1|\beta_j^1)}\mathcal{E}_{\beta_i^1}\mathcal{E}_{\beta_j^1} = \sum_{k\in\mathbb{Z}_+^N} c_k\mathcal{E}^k \tag{5.16}$$

where $c_k \in \mathbb{C}$ *and* $c_k \neq 0$ *only when* $k = (k_1, \dots, k_h)$ *is such that* $k_s = 0$ *for* $s \leq i$ *and* $s \geq j$*, and* $\mathcal{E}^k = \mathcal{E}_{\beta_1^1}^{k_1} \dots \mathcal{E}_{\beta_h^1}^{k_h}$.

(b)

$$\mathcal{F}_{\beta_j}\mathcal{F}_{\beta_i} - \epsilon^{-(\beta_i|\beta_j)}\mathcal{F}_{\beta_i}\mathcal{F}_{\beta_j} = \sum_{k\in\mathbb{Z}_+^N} c_k\mathcal{F}^k \tag{5.17}$$

where $c_k \in \mathbb{C}$ *and* $c_k \neq 0$ *only when* $k = (k_1, \dots, k_N)$ *is such that* $k_s = 0$ *for* $s \leq i$ *and* $s \geq j$*, and* $\mathcal{F}^k = \mathcal{F}_{\beta_N}^{k_N} \dots \mathcal{F}_{\beta_1}^{k_1}$.

As corollary, we conclude that:

Theorem 5.4.17. *The monomials*

$$\mathcal{E}_{\beta_1^1}^{k_1} \dots \mathcal{E}_{\beta_h^1}^{k_h}\mathcal{K}_1^{s_1} \dots \mathcal{K}_n^{s_n}\mathcal{F}_{\beta_N}^{t_1} \dots \mathcal{F}_{\beta_1}^{t_1}$$

for $(k_1, \dots, k_h) \in (\mathbb{Z}^+)^h$, $(t_1, \dots, t_N) \in (\mathbb{Z}^+)^N$ *and* $(s_1, \dots, s_n) \in \mathbb{Z}^n$*, are a* $\mathbb{C}[t]$ *basis of* $\mathcal{U}_\epsilon^t$*. In particular* t *is not a zero divisor in* $\mathcal{U}_\epsilon^t$ *hence* $\mathcal{U}_t^e$ *is a flat over* $\mathbb{C}[t]$

Define $\mathcal{U}^0 = \mathbb{C}[\mathcal{E}_{\beta_1^1}, \mathcal{F}_{\beta_h^1}] \subset \mathcal{S}_\epsilon$, then we can define

$$\mathcal{U}^i = \mathcal{U}_{\sigma,D}^{i-1}\left[\mathcal{E}_{\beta_i^i}, \mathcal{F}_{\beta_{h-i}^1}\right] \subset \mathcal{S}_\epsilon(\mathfrak{p})$$

for $i = 1, \dots, h$. Let $\mathcal{U}^{h+1} = \mathcal{U}^h[\mathcal{F}_{\beta_k^2}]$, then

$$\mathcal{U}^{h+j} = \mathcal{U}_{\sigma,D}^{h+j-1}\left[\mathcal{F}_{\beta_{k-j}^2}\right] \subset \mathcal{S}_\epsilon(\mathfrak{p})$$

for $j = 1, \dots, k$, where σ and D are given by the LS relation. Note now that, the $\mathcal{K}_i$, for $i = 1, \dots, n$ normalize $\mathcal{U}^N$, and when we add them to this algebra we perform an iterated construction of twisted Laurent polynomial. The resulting algebra will be called $\mathcal{T}$.

Lemma 5.4.18.

$$\mathcal{S}_\epsilon(\mathfrak{p}) = \mathcal{T}$$

Proof. Note that, by construction $\mathcal{T} \subset \mathcal{S}_\epsilon$, then we must only demonstrate that $\mathcal{S}_\epsilon \subset \mathcal{A}$. Now note that

$$\mathcal{E}_{\beta_1^1}^{k_1} \cdots \mathcal{E}_{\beta_h^1}^{k_h}\mathcal{K}_1^{s_1} \cdots \mathcal{K}_n^{s_n}\mathcal{F}_{\beta_N}^{t_N} \cdots \mathcal{F}_{\beta_1}^{t_1} \in \mathcal{T}$$

for every $(k_1, \dots, k_h)(\mathbb{Z}^+)^N \in$, $(t_1, \dots, t_N) \in (\mathbb{Z}^+)^N$ and $(s_1, \dots, s_n) \in \mathbb{Z}^n$. Then by proposition 5.4.16 we have $\mathcal{S}_\epsilon \subset \mathcal{T}$. □

Then the claim is proved. □

Theorem 5.4.19. *If l is a good integer (cf theorem 4.5.7)*

$$\deg \mathcal{S}_\epsilon(\mathfrak{p}) = l^{\frac{1}{2}\left(l(w_0)+l\left(w_0^{\mathfrak{l}}\right)+\mathrm{rank}\left(w_0-w_0^{\mathfrak{l}}\right)\right)}$$

Proof. Denote by $\overline{\mathcal{S}}_\epsilon(\mathfrak{p})$ the quasi polynomial algebra associated to $\mathcal{S}_\epsilon(\mathfrak{p})$. We know by the general theory that

$$\deg \mathcal{S}_\epsilon(\mathfrak{p}) = \deg \overline{\mathcal{S}}_\epsilon(\mathfrak{p}).$$

Let x_i denote the class $E_{\beta_i^1}$ for $i = 1, \dots, h$ and y_j the class of F_{β_j} for $j = 1, \dots, N$, then from theorem 5.4.14 we have

$$x_i x_j = \epsilon^{(\beta_i^1|\beta_j^1)} x_j x_i, \tag{5.18}$$

$$y_i y_j = \epsilon^{-(\beta_i|\beta_j)} y_j y_i. \tag{5.19}$$

if $i < j$. Thus we introduce the skew symmetric matrices $A = (a_{ij})$ with $a_{ij} = (\beta_i|\beta_j)$ for $i < j$ and $A^{\mathfrak{l}} = \left(a'_{ij}\right)$ with $a'_{ij} = (\beta_i^1|\beta_j^1)$ for $i < j$.

Let k_i the class of K_i, using the relation in theorem 5.4.14 we obtain a $n \times N$ matrix $B = ((w_i|\beta_j))$ and a $h \times N$ matrix $B^{\mathfrak{l}} = ((w_i|\beta_j^1))$.

Let $t = 2$ unless the Cartan matrix is of type G_2, in which case $t = 6$. Since we will eventually reduce modulo l an odd integer coprime with t, we start inverting t. Thus consider the free $\mathbb{Z}\left[\frac{1}{t}\right]$ module V^+ with basis $u_1, \dots, u_h$, V^- with basis $u'_1, \dots, u'_N$ and V^0 with basis $w_1, \dots, w_n$. On $V = V^+ \oplus V^0 \oplus V^-$ consider the bilinear form given by

$$T = \begin{pmatrix} A^{\mathfrak{l}} & -{}^tB^{\mathfrak{l}} & 0 \\ B^{\mathfrak{l}} & 0 & -B \\ 0 & {}^tB & -A \end{pmatrix},$$

then the rank of T is the degree of $\overline{\mathcal{S}}_\epsilon(\mathfrak{p})$. Consider the operators $M^{\mathfrak{l}} = \begin{pmatrix} A^{\mathfrak{l}} & -{}^tB^{\mathfrak{l}} & 0 \end{pmatrix}$, $M = \begin{pmatrix} 0 & {}^tB & -A \end{pmatrix}$, and $N = \begin{pmatrix} B^{\mathfrak{l}} & 0 & -B \end{pmatrix}$, so that $T = M^{\mathfrak{l}} \oplus N \oplus M$.

Note that

$$B(u'_i) = \beta_i$$

and

$$B^{\mathfrak{l}}(u_i) = \beta_i^1$$

Set $T_1 = M^{\mathfrak{l}} \oplus M$, then using the notation of lemma 4.5.4, we have

Lemma 5.4.20. *The vector $v_\omega = \sum_{t \in I_\omega(w_0^{\mathfrak{l}})} u_t - \omega - w_0(\omega) + \sum_{t \in I_\omega(w_0)} u'_t$, as ω runs through the fundamental weights, form a basis of the kernel of T_1.*

Proof. First, we observe that T_1 is surjective, since M and $M^{\mathfrak{l}}$ are projections over V^- and V^+ respectively, by lemma 4.5.5. Since the n vectors v_ω are part of a basis and, the kernel of T_1 is a direct summand of rank n, by surjectivity. It is enough to show that v_ω is in the kernel of T_1. We have

$$\begin{aligned} T_1(v_\omega) &= A^{\mathfrak{l}}\left(\sum_{t\in I_\omega(w_0^{\mathfrak{l}})} u_t\right) -{}^t B^{\mathfrak{l}}\left(-\omega - w_0(\omega)\right) \\ &\quad +{}^t B\left(-\omega - w_0(\omega)\right) - A\left(\sum_{t\in I_\omega(w_0)} u_t'\right) \\ &= M^{\mathfrak{l}}\left(v_\omega\right) -{}^t B^{\mathfrak{l}}\left(w_0^{\mathfrak{l}}(\omega) - w_0(\omega)\right) - M(v_\omega). \end{aligned}$$

So from lemma 4.5.4 and lemma 4.5.5, we have:

$$T_1(v_w) = -{}^t B^{\mathfrak{l}}\left(w_0^{\mathfrak{l}}(\omega) - w_0(\omega)\right)$$

Let $w_0 = w_0^{\mathfrak{l}}\overline{w}$, since w runs through the fundamental weights, we have two cases:

1. $\overline{w}(\omega) = w$, therefore $w_0^{\mathfrak{l}}(\omega) - w_0(\omega) = 0$ and $T_1(v_\omega) = 0$.
2. $\overline{w}(\omega) \neq \omega$, therefore $w_0^{\mathfrak{l}}(\omega) = \omega$ and $w_0^{\mathfrak{l}}(\omega) - w_0(\omega) = \omega - w_0(\omega) \in \ker {}^tB^{\mathfrak{l}}$, by definition of ${}^tB^{\mathfrak{l}}$, so $T_1(v_\omega) = 0$.

□

Since T is the direct sum of T_1 and N. Its kernel is the intersection of the 2 kernels of these operators. We have computed the kernel of T_1 in 5.4.20. Thus the kernel of T equals the kernel of N restricted to the submodule spanned by the v_ω.

Lemma 5.4.21.

$$N(v_\omega) = \sum_{t\in I_\omega(w_0^{\mathfrak{l}})} \beta_t^1 - \sum_{t\in I_\omega(w_0)} \beta_t = w_0(\omega) - w_0^{\mathfrak{l}}(\omega).$$

Proof. Note that $B(u_t) = \beta_t$, then

$$N(v_w) = \sum_{t\in I_\omega(w_0^{\mathfrak{l}})} \beta_t^1 - \sum_{t\in I_\omega(w_0)} \beta_t. \tag{5.20}$$

Finally, the claim follows using lemma 4.5.4. □

Thus, we can identify N we the map $w_0 - w_0^{\mathfrak{l}} : \Lambda \to Q$. At this point we need the following fact

Lemma 5.4.22. *Let $\theta = \sum_{i=1}^{n} a_i\alpha_i$ the highest root of the root system R. Let $\mathbb{Z}' = \mathbb{Z}[a_1^{-1}, \ldots, a_n^{-1}]$, and let $\Lambda' = \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}'$ and $Q' = Q \otimes_{\mathbb{Z}} \mathbb{Z}'$. Then the $\mathbb{Z}'$ submodule $(w_0 - w_0^{\mathfrak{l}})\Lambda'$ of Q' is a direct summand.*

Proof. The claim follows as a consequence of lemma 4.5.6. □

So if l is a good integer, i.e. l is coprime with t and a_i for all i, we have

$$\operatorname{rank} T = l(w_0) + l(w_0^{\mathfrak{l}}) + n - \Big(n - \operatorname{rank}\Big(w_0 - w_0^{\mathfrak{l}}\Big)\Big),$$

and so the theorem follows. □

5.5 The degree of $\mathcal{U}_\epsilon(\mathfrak{p})$

5.5.1 A family of $\mathcal{U}_\epsilon(\mathfrak{p})$ algebras

As we have seen at the end of section 5.4.1, $Z_0(\mathfrak{p})[t] \subset \mathcal{U}_\epsilon^t(\mathfrak{p})$, so for all $t \in \mathbb{C}$ and $\chi \in \operatorname{Spec}(Z_0(\mathfrak{p}))$, we can define $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p}) = \mathcal{U}_\epsilon^t(\mathfrak{p})/J^\chi$ where J^χ is the two side ideal generated by

$$\ker \chi = \{z - \chi(z) \cdot 1 : z \in Z_0(\mathfrak{p})\}$$

The P.B.W. theorem for $\mathcal{U}_\epsilon^t(\mathfrak{p})$ implies that

Proposition 5.5.1. *The monomials*

$$E_{\beta_1^1}^{s_1} \cdots E_{\beta_h^1}^{s_h} K_1^{r_1} \cdots K_n^{r_n} F_{\beta_h^1}^{t_{k+h}} \cdots F_{\beta_1^1}^{t_{k+1}} F_{\beta_k^2}^{t_k} \cdots F_{\beta_1^2}^{t_1}$$

for which $0 \le s_j, t_i, r_v < l$, form a $\mathbb{C}$ basis for $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$

Lemma 5.5.2. *For every $0 \neq \lambda \in \mathbb{C}$, $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$ is isomorphic to $\mathcal{U}_\epsilon^{\lambda t,\chi}(\mathfrak{p})$.*

Proof. Consider the isomorphism ϑ_λ from $\mathcal{U}_\epsilon^t(\mathfrak{p})$ and $\mathcal{U}_\epsilon^{\lambda t}(\mathfrak{p})$, define by the relation 5.10. Its follows from the above definition that $\vartheta_\lambda(J^\chi) = J^\chi$. Then ϑ_λ induce an isomorphism between $\mathcal{U}_\epsilon^{t,\chi}$ and $\mathcal{U}_\epsilon^{\lambda t,\chi}$. □

Proposition 5.5.3. *The $\mathcal{U}_\epsilon(\mathfrak{p})$ algebras $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$ form a continuous family parameterized by $\mathcal{Z} = \mathbb{C} \times \operatorname{Spec}(Z_0(\mathfrak{p}))$.*

Proof. Let $\mathcal{V}$ denote the set of triple (t, χ, u) with $(t, \chi) \in \mathcal{Z}$ and $u \in \mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$. Then from the P.B.W. theorem we have that the set of monomial

$$E_{\beta_1}^{s_1} \cdots E_{\beta_h^1}^{s_h} K_1^{r_1} \ldots K_n^{r_n} F_{\beta_N}^{t_N} \cdots F_{\beta_1}^{t_1}$$

for which $0 \le s_i, t_i, r_v < l$, for $i \in \Pi^{\mathfrak{l}}$, $j = 1, \ldots, N$ and $v = \ldots, n$, form a basis for each algebra $\mathcal{U}_\epsilon^{t,\chi}$.

Order the previous monomials and assign to $u \in \mathcal{U}_\epsilon^{t,\chi}$ the coordinate vector of u with respect to the ordered basis. This construction identifies $\mathcal{V}$

with $\mathcal{Z} \times \mathbb{C}^d$, where $d = l^{h+n+N}$, thereby giving $\mathcal{A}$ a structure of an affine variety.

Consider the vector bundle $\pi : \mathcal{V} \to \mathcal{Z}$, $(t, \chi, u) \to (t, \chi)$. Note that the structure constant of the algebra $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$, as well as the matrix entries of the linear transformations which define the action of $\mathcal{U}_\epsilon(\mathfrak{p})$ relative to the basis, are polynomial in χ and t. This means that the maps

$$\begin{aligned} \mu : \mathcal{V} \times_{\mathcal{Z}} \mathcal{V} \to \mathcal{V}, &\qquad ((t, \chi, u), (t, \chi, v)) \mapsto (t, \chi, uv) \\ \rho : \mathcal{U}_\epsilon \times \mathcal{V} \to \mathcal{V}, &\qquad (x, (t, \chi, u)) \mapsto (t, \chi, x \cdot u) \end{aligned}$$

where $(t, \chi) \in \mathcal{Z}$, u, $v \in \mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$ and $x \in \mathcal{U}_\epsilon(\mathfrak{p})$, μ defined on $\mathcal{V}$ a structure of vector bundle of algebra and ρ a structure of vector bundle of $\mathcal{U}_\epsilon(\mathfrak{p})$ modules. The fiber of π above (t, χ) is the $\mathcal{U}_\epsilon(\mathfrak{p})$ algebra $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$. □

Note. If we fix $\chi \in \mathrm{Spec}(Z_0)$, we have from theorem 5.4.17 that the family of algebra $\mathcal{U}_\epsilon^{t,\chi}(\mathfrak{p})$ is a flat deformation of algebra over $\mathrm{Spec}\, \mathbb{C}[t]$.

5.5.2 Generically semisimplicity

Summarizing, if ϵ is a primitive l^{th} root of 1 with l odd and $l > d_i$ for all i, we have proven the following facts on $\mathcal{U}_\epsilon$:

- $\mathcal{U}_\epsilon^t$ and $U_\epsilon(\mathfrak{p})$ are domains because $\mathcal{U}_\epsilon(\mathfrak{g})$ it is,
- $\mathcal{U}_\epsilon^t$ and $\mathcal{U}_\epsilon(\mathfrak{p})$ are finite modules over $Z_0[t]$ and Z_0 respectively (cf lemma 5.4.9 and proposition 5.2.2).

Since the L.S. relations holds for $\mathcal{U}_\epsilon(\mathfrak{p})$ and $\mathcal{U}_\epsilon^t$ (cf proposition 5.1.10), we can apply the theory developed in section 4.1.2, and we obtain that $\mathrm{Gr}\,\mathcal{U}_\epsilon(\mathfrak{p})$ and $\mathrm{Gr}\,\mathcal{U}_\epsilon^t$ are twisted polynomial algebra, with some elements inverted. Hence all conditions of theorem 3.5.3 are verified, so

Theorem 5.5.4. *U_ϵ^t and $\mathcal{U}_\epsilon(\mathfrak{p})$ are maximal orders.*

Therefore, $\mathcal{U}_\epsilon(\mathfrak{p}) \in \mathcal{C}_m$, i.e. is an algebra with trace of degree m.

Theorem 5.5.5. *The set*

$$\Omega = \{a \in \mathrm{Spec}(Z_\epsilon(\mathfrak{p})), \textit{ such that the corresponding semisimple representation is irreducible}\}$$

is a Zariski open set. This exactly the part of $\mathrm{Spec}(Z_\epsilon(\mathfrak{p}))$ over which $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ is an Azumaya algebra of degree m.

Proof. Apply theorem 2.2.7, with $R = \mathcal{U}_\epsilon(\mathfrak{p})$ and $T = Z_\epsilon(\mathfrak{p})$. □

Recall that $Z_\epsilon(\mathfrak{p})$ is a finitely generated module over $Z_0(\mathfrak{p})$. Let $\tau : \mathrm{Spec}(Z_\epsilon(\mathfrak{p})) \to \mathrm{Spec}(Z_0(\mathfrak{p}))$ the finite surjective morphism induced by the inclusion of $Z_0(\mathfrak{p})$ in $Z_\epsilon(\mathfrak{p})$. The properness of τ implies the following

Corollary 5.5.6. *The set*

$$\Omega_0 = \{a \in \mathrm{Spec}(Z_0(\mathfrak{p})) : \tau^{-1}(a) \subset \Omega\}$$

is a Zariski dense open subset of $\mathrm{Spec}(Z_0)$.

We know by the theory developed in chapter 3, that $\mathcal{S}_\epsilon(\mathfrak{p}) \in \mathcal{C}_{m_0}$, with $m_0 = l^{l(w_0)+l(w_0^{\mathfrak{l}})+\mathrm{rank}(w_0-w_0^{\mathfrak{l}})}$. As we see in proposition 5.4.9, $S_\epsilon(\mathfrak{p})$ is a finite module over C_0, then C_ϵ, the center of $\mathcal{S}_\epsilon(\mathfrak{p})$ is finite over C_0. The inclusion $C_0 \hookrightarrow C_\epsilon$ induces a projection $\upsilon : \mathrm{Spec}(C_\epsilon) \to \mathrm{Spec}(C_0)$. As before, we have:

Lemma 5.5.7. *(i)*

$$\Omega' = \{a \in \mathrm{Spec}(C_\epsilon), \textit{such that the corresponding semisimple representation is irreducible}\}$$

is a Zariski open set. This exactly the part of $\mathrm{Spec}(C_\epsilon)$ *over which* $\mathcal{S}_\epsilon^\chi(\mathfrak{p})$ *is an Azumaya algebra of degree* m_0.

(ii) The set

$$\Omega_0' = \{a \in \mathrm{Spec}(Z_0(\mathfrak{p})) : \upsilon^{-1}(a) \subset \Omega'\}$$

is a Zariski dense open subset of $\mathrm{Spec}(Z_0)$.

Proof. Apply theorem 2.2.7 at $\mathcal{S}_\epsilon(\mathfrak{p})$. □

Note. Since $\mathrm{Spec}(Z_0)$ is irreducible, we have that $\Omega_0 \cap \Omega_0'$ is non empty.

We can state the main theorem of this section

Theorem 5.5.8.

$$\deg \mathcal{U}_\epsilon(\mathfrak{p}) = l^{\frac{1}{2}\left(l(w_0)+l(w_0^{\mathfrak{l}})+\mathrm{rank}\left(w_0-w_0^{\mathfrak{l}}\right)\right)}$$

Proof. For $\chi \in \Omega_0 \cap \Omega_0'$, we have, using theorem 5.5.5 and lemma 5.5.7,

$$\deg \mathcal{U}_\epsilon(\mathfrak{p}) = m = \deg \mathcal{U}_\epsilon^\chi(\mathfrak{p}),$$

and

$$\deg \mathcal{S}_\epsilon^\chi(\mathfrak{p}) = \deg \mathcal{S}_\epsilon(\mathfrak{p}).$$

But for all $t \neq 0$, we have that $\mathcal{U}_\epsilon^{t,\chi}$ is isomorphic to $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ as algebra. Hence is well known that the isomorphism class of semisimple algebras are closed (see [Pro98] or [Pie82]), we have that $\mathcal{S}_\epsilon^\chi(\mathfrak{p}) = \mathcal{U}_\epsilon^{0,\chi}$ is isomorphic to $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$. Then

$$\deg \mathcal{U}_\epsilon(\mathfrak{p}) = m = \deg \mathcal{U}_\epsilon^\chi(\mathfrak{p}) = \deg \mathcal{S}_\epsilon^\chi(\mathfrak{p}) = \deg \mathcal{S}_\epsilon(\mathfrak{p}).$$

And by theorem 5.4.19 the claim follows. □

Note. As we see in section 3.5, since $\mathcal{U}_\epsilon(\mathfrak{p})$ is a maximal order, $Z_\epsilon(\mathfrak{p})$ is integrally closed, so following example 2.2.4, we can make the following construction: denote by $Q_\epsilon := Q(Z_\epsilon(\mathfrak{p}))$ the field of fractions of $Z_\epsilon(\mathfrak{p})$, we have that $Q(\mathcal{U}_\epsilon(\mathfrak{p})) = \mathcal{U}_\epsilon(\mathfrak{p}) \otimes_{Z_\epsilon(\mathfrak{p})} Q_\epsilon$ is a division algebra, finite dimensional over its center Q_ϵ. Denote by $\mathcal{F}$ the maximal commutative subfield of $Q(\mathcal{U}_\epsilon(\mathfrak{p}))$, we have, using standard tool of associative algebra (cf [Pie82]), that

(i) $\mathcal{F}$ is a finite extension of Q_ϵ of degree m,

(ii) $Q(\mathcal{U}_\epsilon(\mathfrak{p}))$ has dimension m^2 over Q_ϵ,

(iii) $Q(\mathcal{U}_\epsilon(\mathfrak{p})) \otimes_{Q_\epsilon} \mathcal{F} \cong M_m(\mathcal{F})$.

Hence, we have that

$$\begin{aligned} \dim_{Q(Z_0(\mathfrak{p}))} Q_\epsilon &= \deg \tau \\ \dim_{Q_\epsilon} Q(\mathcal{U}_\epsilon(\mathfrak{p})) &= m^2 \\ \dim_{Q(Z_0(\mathfrak{p}))} Q(\mathcal{U}_\epsilon(\mathfrak{p})) &= l^{h+N+n} \end{aligned}$$

where, the first equality is a definition, the second has been pointed out above and the third follows from the P.B.W theorem. Then, we have

$$l^{h+N+n} = m^2 \deg \tau$$

with $m = l^{\frac{1}{2}\left(l(w_0)+l(w_0^{\mathfrak{l}})+\mathrm{rank}\left(w_0-w_0^{\mathfrak{l}}\right)\right)}$, so

Corollary 5.5.9.

$$\deg \tau = l^{n-\mathrm{rank}\left(w_0-w_0^{\mathfrak{l}}\right)}.$$

5.5.3 The center of $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$

To conclude we want to explain a method, inspired by the work of Premet and Skryabin ([PS99]), which in principle allows us to determine the center of $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ for all $\chi \in \mathrm{Spec}(Z_0)$.

Let $\chi_0 \in \mathrm{Spec}(Z_0(\mathfrak{p}))$ define by $\chi_0(E_i) = 0$, $\chi_0(F_i) = 0$ and $\chi_0(K_i^{\pm 1})$, we set $\mathcal{U}_\epsilon^0(\mathfrak{p}) = \mathcal{U}_\epsilon^{\chi_0}(\mathfrak{p})$.

Proposition 5.5.10. *$\mathcal{U}_\epsilon^0(\mathfrak{p})$ is a Hopf algebra with the comultiplication, counit and antipode induced by $\mathcal{U}_\epsilon(\mathfrak{p})$.*

Proof. This is immediately since J^{χ_0} is an Hopf ideal. □

Proposition 5.5.11. *let $\chi \in \mathrm{Spec}(Z_0(\mathfrak{p}))$*

(i) $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ is an $\mathcal{U}_\epsilon(\mathfrak{p})$ module, with the action define by

$$a \cdot u = \sum a_{(1)} u S(a_{(2)}),$$

where $\Delta(a) = \sum a_{(1)} \otimes a_{(2)}$.

(ii) $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ *is an* $\mathcal{U}_\epsilon^0(\mathfrak{p})$ *module, with the action induced by* $\mathcal{U}_\epsilon(\mathfrak{p})$.

Proof. (i) We must verify the relation 4.2.2 on the generators. For any $u \in \mathcal{U}_\epsilon^\chi$, we have

$$\begin{aligned} E_i \cdot u &= E_i u + K_i u S(E_i) = E_i u - K_i u K_i^{-1} E_i, && (5.21a)\\ F_i \cdot u &= F_i u K_i - u F_i K_i, && (5.21b)\\ K_i \cdot u &= K_i u K_i^{-1}. && (5.21c)\\ & && (5.21d)\end{aligned}$$

Then

$$\begin{aligned} [E_i, F_i] \cdot u &= E_i F_i \cdot u - F_i E_i \cdot u \\ &= [E_i, F_i]\, u S\left(K_i^{-1}\right) + K_i u S\left([E_i, F_i]\right) \\ &= L_i u S\left(K_i^{-1}\right) + K_i u S\left(L_i\right) \\ &= L_i \cdot u. \end{aligned}$$

Now we verify the ϵ-Serre relation in the case of $a_{ij} = -1$, we have:

$$\begin{gathered} E_i^2 E_j \cdot u = E_i^2 E_j u + K_i^2 K_j u S(E_i^2 E_j) \\ +E_i^2 K_j u S(E_j) + (1+\epsilon_i^{-2}) K_i E_i E_j u S(E_i) \\ +(1+\epsilon_i^{-2}) K_i E_i K_j u S(E_i E_j) + K_i^2 E_j u S(E_i^2), \end{gathered} \tag{5.22}$$

$$\begin{gathered} E_i E_j E_i \cdot u = E_i E_j E_i u + K_i E_j E_i u S(E_i) + E_i K_j E_i u (S(E_j) \\ +K_i K_j E_i u S(E_i E_j) + E_i E_j K_i u S(E_i) \\ +K_i E_j K_i u S(E_i^2) + Ei K_j K_i u S(E_j E_i) + K_i^2 K_j u S(E_i E_j E_i), \end{gathered} \tag{5.23}$$

$$\begin{gathered} E_j E_i^2 \cdot u = E_j E_i^2 u + K_j E_i^2 u S(E_j) \\ +(1+\epsilon_i^{-2}) E_j K_i E_i u S(E_i) + (1+\epsilon_i^{-2}) K_j K_i E_i u S(E_j E_i) \\ +E_j K_i^2 u S(E_i^2) + K_j K_i^2 u S(E_j E_i^2). \end{gathered} \tag{5.24}$$

Note that $1 + \epsilon_i^2 - [2]_{d_i} \epsilon_i^{-1} = 0$, where $[n]_{d_i} = \frac{q^n - q^{-n}}{\epsilon^{d_i} - \epsilon^{-d_i}}$, then

$$\left(E_i^2 E_j - [2]_{d_i} E_i E_j E_i + E_j E_i^2\right) \cdot u = 0$$

All other relation can be obtain with similar calculus. So $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ is an $\mathcal{U}_\epsilon(\mathfrak{p})$ module.

(ii) Recall that E_i^l, F_j^l, and $K_j^{\pm l}$ are in the center of $\mathcal{U}_\epsilon(\mathfrak{p})$ for $i \in \Pi^{\mathfrak{l}}$ and $j = 1, \ldots, n$, and that

$$\begin{aligned} \Delta(E_i^l) &= E_i^l \otimes 1 + K_i^l \otimes E_i^l \\ \Delta(F_i^l) &= F_i^l \otimes K_i^{-l} + 1 \otimes F_i^l \\ \Delta(K_i^{\pm l}) &= K_i^{\pm l} \otimes K_i^{\pm l} \end{aligned}$$

Then

$$\begin{aligned}
E_i^l \cdot u &= E_i^l u - K_i^l u K_i^{-l} E_i^l = 0, \\
F_i^l \cdot u &= F_i^l u K_i^{-l} - u F_i^l K_i^{-l} = 0, \\
K_i^l \cdot u &= K_i^l u K_i^{-l} = u.
\end{aligned}$$

It follows that $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ is an $\mathcal{U}_\epsilon^0(\mathfrak{p})$ module.

□

Proposition 5.5.12. *Let $x \in \mathcal{U}_\epsilon^\chi(\mathfrak{p})$, then x is in the center of $\mathcal{U}_\epsilon^\chi(\mathfrak{p})$ if and only if x is invariant under the action of $\mathcal{U}_\epsilon^0(\mathfrak{p})$, i.e.*

$$E_i \cdot x = 0, \tag{5.25a}$$
$$F_i \cdot x = 0, \tag{5.25b}$$
$$K_i \cdot x = x. \tag{5.25c}$$

Proof. Let $x \in Z\left(\mathcal{U}_\epsilon^\chi(\mathfrak{p})\right)$ then

$$E_i \cdot x = E_i x - K_i x K_i^{-1} E_i = 0,$$

in the same way we obtain the other relations.

Suppose now that x verify the relations 5.25. Then

$$K_i \cdot x = K_i x K_i^{-1} = x,$$

imply that $K_i x = x K_i$. From $E_i \cdot x = 0$ we obtain

$$\begin{aligned}
0 = E_i \cdot x &= E_i x - K_i x K_i^{-1} E_i \\
&= E_i x - x E_i.
\end{aligned}$$

its follows that $E_i x = x E_i$. In the same way we have $F_i x = x F_i$. Then x lies in the center. □

So we can determine the center at t generic by lifting the center of the algebra at $t = 0$. We obtain an analogue of the Harish Chandra theory for semisimple Lie algebras.

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